Insights into the Theory of Relativity.
Part II.
Lorentz Transformation.

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Abstract

In this article we re-analyse the orthogonal endomorphisms structure. This is because of we prove previously (in Part I) that skew-adjoint endomorphism structure holds invariant in an orthogonal transformation. After that we infer the Lorentz boost and 2-dimensional euclidian rotation highlighting their geometrical structure in the context of annihilating polynomials of the mentioned skew-adjoint endomorphisms, namely electromagnetic field associated endomorphisms, and orthogonal endomorphism as well. This has meaningful involvements inside the structure of the base of special theory of relativity.

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Contents

1 Introduction. Part II.

In the previous paper [1], (see [7]) it was proved that an orthogonal homomorphism R preserves the skew-adjoint structures of the skew-adjoint endomorphisms associated to the electromagnetic fields tensors of two observers that observe the same event from the space-time of each one.

In the aforementioned paper ([7]) it was proved also that space-times of both observers are boiled down to only one space-time. This is because of, for the moment, the nature of the affine structure of the space into which are immersed the spaces-times of both observers. Therefore we deem that R acts now as an orthogonal endomorphism into $\mathbb{L}_O$. Namely R is the Lorentz transformation [2]. $\mathbb{L}_O$ is the vectorial lorentzian space-time of the observer. As a matter of convenience henceforth we symbolize the space-time by $\mathbb{L}_4$ instead of $\mathbb{L}_O$.

In this paper our primary purpose is the study of two types of orthogonal endomorphisms we select. They are the two most relevant and meaningful types of transformations of the early stage of the relativity theory namely Lorentz boost and 2-dimensional euclidean rotation as well.

There are some other types of orthogonal transformations on $\mathbb{L}_4$ usually identities, reflections,etc.. and other singular or special cases. It is worthwhile to deal with them aside.

Thereafter in Section 4 we develop the two aforementioned types of orthogonal endomorphisms, that is to say Lorentz boost and 2 dimensional euclidean rotation highlighting his geometrical structure [3] fitted to the structure of skew-adjoint endomorphism F (that is electromagnetic field) that I worked out in my paper "Structures of the Skew-adjoint Endomorphisms and Some Peculiarities of Electromagnetic Field.(2014)" (see it in www.relativityworkshop.com or [6]).

Outcomes we have gotten for electromagnetic case (see [6]) and the types of orthogonal endomorphisms selected in this paper are adapted to relativity theory we analyze here.

In this paper it is relevant to single out the known importance of electromagnetic field in the theory of relativity. Poincaré inferred Lorentz transformation searching for transformations that preserve Maxwell equations. Einstein got it by mean other ways but de-

[1] This Part I, is titled "Insight into relativity theory. Part I. Critical approach. Basic principles and starting points.

[2] see [7] sections 3.2 y 4.

[3] This is one of the more relevant contributions of this paper.

veloped the physical meaning and interpretations of Lorentz transformations. Anyway the classical relativity theory rest upon how electromagnetic signals and interactions work.

The electromagnetic field is so attached to the theory of relativity that the theory of relativity is to be constructed out of the electromagnetic field structure [4].

The most of physical events that are happening in the macrocosm are for reason of electromagnetic phenomena (together other phenomena as quantum phenomena, etc.. without thinking into gravity). For this reason electromagnetic fields penetrate deeply into the mechanics of the physical world. We work on the basis of these concepts.

It is worthwhile to single out that the contraction-dilatation of space and time rest upon the measurements of electromagnetic signals, usually light signals. In an hypothetical case in which signals could belong to other field -not the electromagnetic but rather other fields that supposedly would affect the most of phenomena of our macrocosm - it is sure that we could have other theory of relativity different.

2 Condition of orthogonality.

In accordance with the Part I sections 3.1.2. and 4. we work on a vectorial lorentzian space-time in which $G = G_1 = G_2$ as a result of the *transporter principle* that we explain in Part I, section 3.2. .

Therefore the orthogonality condition of an endomorphism R becomes

$$G = RGR^t$$

Actually R is the Lorentz transformation. We are working on inertial reference frames taking into account that for the moment we are fitted to the context of special relativity.

3 Invariant subspaces decomposition in skew-adjoint endomorphisms.

Taking into account that the electromagnetic field F is represented by his associated endomorphism into $\mathbb{L}_4$ we shall study F as an skew-adjoint endomorphism.

[4]Also taking into regard several classical principles and postulates for example relativity principle, isotropy and homogeneity of space, the timing of clocks, and so on.

According with my paper [6] the annihilating polynomial of a skew-adjoint endomorphism F in a lorentzian space-time is:

$$(F^2 - \lambda^2 I)(F^2 + \mu^2 I) = 0$$
$$\lambda \neq 0 \text{ or } \mu \neq 0$$

In view of the mentioned decomposition, **the orthogonal endomorphism R leaves invariant the same subspaces than F.** These invariants subspaces are $\mathbb{L}_2$ and $\mathbb{E}_2$. $R_{\mathbb{L}}$ acts into $\mathbb{L}_2$ and $R_{\mathbb{E}}$ acts into $\mathbb{E}_2$.

$R_{\mathbb{L}}$ and $R_{\mathbb{E}}$ are the endomorphisms that make up R [5].

4 Orthogonal transformation. Subspaces invariant in orthogonal endomorphisms.

Hereupon this paper is devoted to the analysis of the orthogonal endomorphism R in order to derive the Lorentz transformation (specially the Lorentz boost).

In view of the above sections, in this section we only analyse the 2-dim orthogonal endomorphism $R_{\mathbb{L}}$ on $\mathbb{L}_2$ with minimal polynomial $P_{\mathbb{L}}(R_{\mathbb{L}}) \equiv (R_{\mathbb{L}} - \lambda_p I)(R_{\mathbb{L}} - \lambda_q I) = 0$; ($\lambda_p \neq 0$ or $\lambda_q \neq 0$ and $\lambda_p \neq \lambda_q$) and 2-dim orthogonal endomorphism $R_{\mathbb{E}}$ on $\mathbb{E}_2$ with minimal polynomial $P_{\mathbb{E}}(R_{\mathbb{E}}) \equiv R_{\mathbb{E}}^2 + 2a_1 R_{\mathbb{E}} + a_0 I$ being $P_{\mathbb{E}}(R_{\mathbb{E}})$ irreducible (we shall prove further along $a_0 = 1$). For the moment we are only interested in these two cases. Further orthogonal types are put aside for his afterward study in other paper. We are thinking of this orthogonal types selection abiding by our conclusions in [6]).

4.1 Case of the minimal polynomial $P_{\mathbb{L}}(R_{\mathbb{L}}) \equiv (R_{\mathbb{L}} - \lambda_p I)(R_{\mathbb{L}} - \lambda_q I) = 0$ on $\mathbb{L}_2$.

In this subsection we begin analysing eigenvectors and eigenvalues of orthogonal endomorphism $R_{\mathbb{L}}$ on $\mathbb{L}_2$, that is to say in the straightforward case in which the annihilating polynomial on $\mathbb{L}_2$ is

$$P_{\mathbb{L}}(R_{\mathbb{L}}) \equiv (R_{\mathbb{L}} - \lambda_p I)(R_{\mathbb{L}} - \lambda_q I), \lambda_p \neq 0 \text{ or } \lambda_q \neq 0 \text{ and } \lambda_p \neq \lambda_q$$

Therefore we have:

$$R_{\mathbb{L}}(\overrightarrow{\mathbf{p}}) = \lambda_p \overrightarrow{\mathbf{p}}$$
$$R_{\mathbb{L}}(\overrightarrow{\mathbf{q}}) = \lambda_q \overrightarrow{\mathbf{q}}$$

[5] In fact R must preserve the metric lorentzian into $\mathbb{L}_2$ and the metric euclidean into $\mathbb{E}_2$ independently.

$\overrightarrow{\mathbf{p}}$ and $\overrightarrow{\mathbf{q}}$ are eigenvectors.

Because of orthogonality

$$R_{\mathbb{L}}(\overrightarrow{\mathbf{p}}).R_{\mathbb{L}}(\overrightarrow{\mathbf{p}}) = \overrightarrow{\mathbf{p}}^2 = {\lambda_p}^2 \overrightarrow{\mathbf{p}}^2 \tag{1}$$

$$R_{\mathbb{L}}(\overrightarrow{\mathbf{q}}).R_{\mathbb{L}}(\overrightarrow{\mathbf{q}}) = \overrightarrow{\mathbf{q}}^2 = {\lambda_q}^2 \overrightarrow{\mathbf{q}}^2 \tag{2}$$

$$R_{\mathbb{L}}(\overrightarrow{\mathbf{p}}).R_{\mathbb{L}}(\overrightarrow{\mathbf{q}}) = \overrightarrow{\mathbf{p}}.\overrightarrow{\mathbf{q}} = \lambda_p \lambda_q \overrightarrow{\mathbf{p}}.\overrightarrow{\mathbf{q}} \tag{3}$$

A solution is: $\overrightarrow{\mathbf{p}}^2 = \overrightarrow{\mathbf{q}}^2 = 0$; $\lambda_p.\lambda_q = 1$.

$\mathbb{L}_2$ is lorentzian because it has two null vectors. For this reason we have $\mathbb{L}_2 \equiv L_2$.

Henceforth (inside this subsection 4.1) we will deal with this solution unless otherwise specified.

Summing up we have in this context two null eigenvectors of R_L $\overrightarrow{\mathbf{p}}$ and $\overrightarrow{\mathbf{q}}$ with eigenvalues λ_p and λ_q.

In this context, on L_2 the minimal polynomial of R_L is

$$P_L(R_L) = (R_L - \lambda_p I)(R_L - \frac{1}{\lambda_p} I); \lambda_{p \neq 0}$$

The matrix components of R_L is

$$(R_L) = \begin{pmatrix} \lambda_p & 0 \\ 0 & \lambda_q \end{pmatrix}; \lambda_p.\lambda_q = 1$$

in the reference frame $(\overrightarrow{\mathbf{p}}, \overrightarrow{\mathbf{q}})$.

4.1.1 Deduction of Lorentz boost.

Herein we make a change of base from the base $(\overrightarrow{\mathbf{p}}, \overrightarrow{\mathbf{q}}, \overrightarrow{\mathbf{Y}}, \overrightarrow{\mathbf{Z}})$ to the base $(\overrightarrow{\mathbf{U}}, \overrightarrow{\mathbf{X}}, \overrightarrow{\mathbf{Y}}, \overrightarrow{\mathbf{Z}})$

$$\overrightarrow{\mathbf{p}}^2 = \overrightarrow{\mathbf{q}}^2 = 0; \overrightarrow{\mathbf{p}}.\overrightarrow{\mathbf{q}} = w$$

$$\overrightarrow{\mathbf{X}}.\overrightarrow{\mathbf{Y}} = \overrightarrow{\mathbf{Y}}.\overrightarrow{\mathbf{Z}} = \overrightarrow{\mathbf{X}}.\overrightarrow{\mathbf{Z}} = \overrightarrow{\mathbf{U}}.\overrightarrow{\mathbf{X}} = \overrightarrow{\mathbf{U}}.\overrightarrow{\mathbf{Y}} = \overrightarrow{\mathbf{U}}.\overrightarrow{\mathbf{Z}} = 0$$

$$\overrightarrow{\mathbf{p}}.\overrightarrow{\mathbf{Y}} = \overrightarrow{\mathbf{p}}.\overrightarrow{\mathbf{Z}} = \overrightarrow{\mathbf{q}}.\overrightarrow{\mathbf{Y}} = \overrightarrow{\mathbf{q}}.\overrightarrow{\mathbf{Z}} = 0$$

$$\overrightarrow{\mathbf{X}}^2 = \overrightarrow{\mathbf{Y}}^2 = \overrightarrow{\mathbf{Z}}^2 = 1; \overrightarrow{\mathbf{U}}^2 = -1$$

The equations of change of base are:

$$\overrightarrow{\mathbf{p}} = a(\overrightarrow{\mathbf{U}} + \overrightarrow{\mathbf{X}})$$

$$\overrightarrow{\mathbf{q}} = b(-\overrightarrow{\mathbf{U}} + \overrightarrow{\mathbf{X}})$$

$$a > 0; b < 0$$

(see ANNEX B.2)

Hence the matrix components of the change of base of $(\overrightarrow{\mathbf{p}}, \overrightarrow{\mathbf{q}})$ to $(\overrightarrow{\mathbf{U}}, \overrightarrow{\mathbf{X}})$ is

$$T = \begin{pmatrix} a & a \\ -b & b \end{pmatrix}$$

The orthogonal endomorphism matrix components (on L_2) on the base $(\overrightarrow{\mathbf{p}}, \overrightarrow{\mathbf{q}})$ is

$$(R_L) = \begin{pmatrix} \lambda_p & 0 \\ 0 & \lambda_q \end{pmatrix}; \lambda_p.\lambda_q = 1$$

Changing the base from reference $(\overrightarrow{\mathbf{p}}, \overrightarrow{\mathbf{q}})$ to the base $(\overrightarrow{\mathbf{U}}, \overrightarrow{\mathbf{X}})$ the components matrix endomorphism is:

$$(R'_L) = (T^{-1})(R_L)(T) = \frac{1}{2ab} \begin{pmatrix} b & -a \\ b & a \end{pmatrix} \begin{pmatrix} \lambda_p & 0 \\ 0 & \lambda_q \end{pmatrix} \begin{pmatrix} a & a \\ -b & b \end{pmatrix};$$

$$\lambda_p.\lambda_q = 1$$

Therefore

$$(R'_L) = \frac{1}{2} \begin{pmatrix} \lambda_p + \lambda_q & \lambda_p - \lambda_q \\ \lambda_p - \lambda_q & \lambda_p + \lambda_q \end{pmatrix}; \lambda_p.\lambda_q = 1$$

We make

$$\frac{1}{2}(\lambda_p + \lambda_q) = \cosh\phi$$

$$\frac{1}{2}(\lambda_p - \lambda_q) = \sinh\phi$$

Needless to say that $(\cosh\phi)^2 - (\sinh\phi)^2 = 1$ taking into account $\lambda_p.\lambda_q = 1$.

Hence **the Lorentz boost transformation is**

$$\boxed{(R'_L) = \begin{pmatrix} \cosh\phi & \sinh\phi \\ \sinh\phi & \cosh\phi \end{pmatrix}}$$

Further to that we can derive the invariants of de Lorentz boost as function of ϕ .

$$\lambda_p = \cosh\phi + \sinh\phi$$

$$\lambda_q = \cosh\phi - \sinh\phi$$

It is derived

$$\boxed{\lambda_p = e^{\phi}} \quad (4)$$

$$\boxed{\lambda_q = e^{-\phi}} \tag{5}$$

It is clear that keeping in mind that $\lambda_p.\lambda_q = 1$

$$\lambda_p^2 - \lambda_q^2 = 2\sinh 2\phi$$
$$\lambda_p^2 + \lambda_q^2 = 2\cosh 2\phi$$

It is not hard to see -reasoning likewise classical relativity-

$$\boxed{\sinh\phi = \frac{-\dfrac{v}{c}}{\sqrt{1-\dfrac{v^2}{c^2}}}}$$

$$\boxed{\cosh\phi = \frac{1}{\sqrt{1-\dfrac{v^2}{c^2}}}}$$

where $\overrightarrow{\mathbf{v}}$ is the velocity of a reference frame respect other reference frame at rest and $v = \sqrt{\overrightarrow{\mathbf{v}}^2}$.

Thereby the invariant in the Lorentz boost are:

$$\lambda_p = \sqrt{\frac{1+\dfrac{v}{c}}{1-\dfrac{v}{c}}}$$

$$\lambda_q = \sqrt{\frac{1-\dfrac{v}{c}}{1+\dfrac{v}{c}}}$$

We have that the Lorentz boost involves a direction $\overrightarrow{\mathbf{X}}$ and the velocity $\overrightarrow{\mathbf{v}}$ of reference frame in the direction $\overrightarrow{\mathbf{X}}$.

4.2 Case in which orthogonal endomorphisms have a second grade irreducible minimal polynomial on $\mathbb{E}_2$.Euclidean 2-dim rotation.

In this subsection we continue with other case analysing the orthogonal endomorphism $R_{\mathbb{E}}$ on $\mathbb{E}_2$. The annihilating polynomial of $R_{\mathbb{E}}$ that concern us is $P_{\mathbb{E}}(R_{\mathbb{E}}) \equiv R_{\mathbb{E}}^2 + 2a_1R_{\mathbb{E}} + a_0I$ where $P_{\mathbb{E}}(R_{\mathbb{E}})$ is irreducible. In this way we can undertake the analysis of the rotation on the subspace $\mathbb{E}_2$ after in this subsection.

In $\mathbb{E}_2$ we rule out the following cases of annihilating polynomials:

$$P_{\mathbb{E}}(R_{\mathbb{E}}) = (R_{\mathbb{E}} - \lambda_1)(R_{\mathbb{E}} - \lambda_2) \tag{6}$$

see [6].

$$P_{\mathbb{E}}(R_{\mathbb{E}}) = (R_{\mathbb{E}} - \lambda)^2$$

On $\mathbb{E}_2$ we only deal with

$$P_{\mathbb{E}}(R_{\mathbb{E}}) = R_{\mathbb{E}}^2 + 2a_1 R_{\mathbb{E}} + a_0 I$$

being $R_{\mathbb{E}}^2 + 2a_1 R_{\mathbb{E}} + a_0 I$ irreducible.

4.2.1 Algebraic structure of irreducible polynomial $R_{\mathbb{E}}^2 + 2a_1 R_{\mathbb{E}} + a_0 I$.

In the following lines we shall prove:

In the annihilating polynomial

$$P_{\mathbb{E}}(R_{\mathbb{E}}) = R_{\mathbb{E}}^2 + 2a_1 R_E + a_0 I$$

($P_{\mathbb{E}}(R_{\mathbb{E}})$ irreducible) if $R_{\mathbb{E}}$ is orthogonal then $a_0 = 1$.

Taking into account that

$$\forall \overrightarrow{\mathbf{X}}, \overrightarrow{\mathbf{X}} \in \mathbb{E}_2; R_{\mathbb{E}}(\overrightarrow{\mathbf{X}}).R_{\mathbb{E}}(\overrightarrow{\mathbf{X}}) = \overrightarrow{\mathbf{X}}.\overrightarrow{\mathbf{X}}$$

$$R_{\mathbb{E}}^2(\overrightarrow{\mathbf{X}}).R_{\mathbb{E}}^2(\overrightarrow{\mathbf{X}}) = \overrightarrow{\mathbf{X}}.\overrightarrow{\mathbf{X}}$$

$$R_{\mathbb{E}}^2(\overrightarrow{\mathbf{X}}).R_{\mathbb{E}}(\overrightarrow{\mathbf{X}}) = R_E(\overrightarrow{\mathbf{X}}).\overrightarrow{\mathbf{X}}$$

we have

$$R_{\mathbb{E}}^2(\overrightarrow{\mathbf{X}}).P_{\mathbb{E}}(R_{\mathbb{E}})\overrightarrow{\mathbf{X}} = \overrightarrow{\mathbf{X}}.\overrightarrow{\mathbf{X}} + 2a_1 R_{\mathbb{E}}(\overrightarrow{\mathbf{X}}).\overrightarrow{\mathbf{X}} + a_0 R_{\mathbb{E}}^2(\overrightarrow{\mathbf{X}}).\overrightarrow{\mathbf{X}} = 0 \tag{7}$$

$$R_{\mathbb{E}}(\overrightarrow{\mathbf{X}}).P_{\mathbb{E}}(R_E)\overrightarrow{\mathbf{X}} = \overrightarrow{\mathbf{X}}.R_{\mathbb{E}}(\overrightarrow{\mathbf{X}}) + 2a_1 \overrightarrow{\mathbf{X}}.\overrightarrow{\mathbf{X}} + a_0 \overrightarrow{\mathbf{X}}.R_{\mathbb{E}}(\overrightarrow{\mathbf{X}}) = 0 \tag{8}$$

$$\overrightarrow{\mathbf{X}}.P_{\mathbb{E}}(R_E)\overrightarrow{\mathbf{X}} = \overrightarrow{\mathbf{X}}.R_{\mathbb{E}}^2(\overrightarrow{\mathbf{X}}) + 2a_1 \overrightarrow{\mathbf{X}}.R_{\mathbb{E}}(\overrightarrow{\mathbf{X}}) + a_0 \overrightarrow{\mathbf{X}}.\overrightarrow{\mathbf{X}} = 0 \tag{9}$$

Subtracting 9 from 7 it is inferred

$$a_0 = 1$$

Furthermore from 8 it is inferred

$$\forall \overrightarrow{\mathbf{X}} \qquad \overrightarrow{\mathbf{X}} \in \mathbb{E}_2; \overrightarrow{\mathbf{X}}.A.\overrightarrow{\mathbf{X}} = 0 \tag{10}$$

where

$$A = R_E + a_1 I \tag{11}$$

That means that A **is skew-adjoint**.

[6] however in section 4.3 "Types of Lorentz transformation", we shall accept $\lambda_1 = \lambda_2$ in 6 in order to define de pure Lorentz transformation. In this case $P_{\mathbb{E}}(R_{\mathbb{E}})$ is not minimal polynomial.

4.2.2 Analysis of the euclidean rotation. Euclidean feature of $\mathbb{E}_2$.; $\mathbb{E}_2 \equiv E_2$.

As $R_{\mathbb{E}}^2 + 2a_1 R_{\mathbb{E}} + a_0 I$ is irreducible it is not hard to check that $a_1 < 1$.

Regarding $a_0 = 1$ as we saw before we have

$$(R_{\mathbb{E}}) = \begin{pmatrix} 0 & 1 \\ -1 & -2a_1 \end{pmatrix}$$

with respect to a cyclic base $(\overrightarrow{\mathbf{Y}}, \overrightarrow{\mathbf{Z}})$

We have:

$$R_{\mathbb{E}}(\overrightarrow{\mathbf{Y}}) = \overrightarrow{\mathbf{Z}}$$

$$R_{\mathbb{E}}(\overrightarrow{\mathbf{Z}}) = -\overrightarrow{\mathbf{Y}} - 2a_1 \overrightarrow{\mathbf{Z}}$$

For reason of $R_{\mathbb{E}}$ is orthogonal:

$$R_{\mathbb{E}}(\overrightarrow{\mathbf{Y}}).R_{\mathbb{E}}(\overrightarrow{\mathbf{Y}}) = (\overrightarrow{\mathbf{Y}})^2 = (\overrightarrow{\mathbf{Z}})^2$$

$$R_{\mathbb{E}}(\overrightarrow{\mathbf{Z}}).R_{\mathbb{E}}(\overrightarrow{\mathbf{Z}}) = (\overrightarrow{\mathbf{Z}})^2 = (\overrightarrow{\mathbf{Y}} + 2a_1 \overrightarrow{\mathbf{Z}})^2$$

$$R_{\mathbb{E}}(\overrightarrow{\mathbf{Y}}).R_{\mathbb{E}}(\overrightarrow{\mathbf{Z}}) = \overrightarrow{\mathbf{Y}}.\overrightarrow{\mathbf{Z}} = -\overrightarrow{\mathbf{Y}}.\overrightarrow{\mathbf{Z}} - 2a_1 \overrightarrow{\mathbf{Z}}^2$$

Let be now

$$(\overrightarrow{\mathbf{Y}})^2 = \mathbf{g}_{22}; (\overrightarrow{\mathbf{Z}})^2 = \mathbf{g}_{33}; \overrightarrow{\mathbf{Y}}\overrightarrow{\mathbf{Z}} = \mathbf{g}_{23}$$

It is derived

$$(\mathbf{G}) = \begin{pmatrix} g_{22} & -a_1 g_{22} \\ -a_1 g_{22} & g_{22} \end{pmatrix}$$

In the base $(\overrightarrow{\mathbf{Y}}, \overrightarrow{\mathbf{Z}})$

$$det\mathbf{G} = g_{22}^2(1 - a_1^2)) > 0$$

since $\mid a_1 \mid < 1$. On the basis of Sylvester theorem we can find a change of base in such a way that the new base has the metric

$$(\mathbf{G}) = \begin{pmatrix} \varepsilon & 0 \\ 0 & 1 \end{pmatrix}, \varepsilon = \pm 1$$

Because of the invariance of $det\mathbf{G}$ is to be $\varepsilon = +1$. Thereby $\mathbb{E}_2 \equiv E_2$ $R_{\mathbb{E}} \equiv R_E$ and $P_{\mathbb{E}} \equiv P_E$ [7].

[7] $\mathbb{E}_2$ is in general a space of any type; we use the type symbol E_2 whenever space is euclidean; in general black bold letter means any type.

4.2.3 Deduction of the euclidean rotation.

Let be now $A_1 = a_1 R_E + I$;taking into account 10 it is clear that $P_E(R_E) = A.R_E + A_1$.

We select an orthonormal base. In the 2-dim eucledian space that concern us the skew-adjoint endomorphism A in this orthonormal base is:

$$A = \begin{pmatrix} 0 & b_1 \\ -b_1 & 0 \end{pmatrix} \tag{12}$$

Therefore taking into account 10 and 12

$$R_E = A - a_1 I = \begin{pmatrix} -a_1 & b_1 \\ -b_1 & -a_1 \end{pmatrix} \tag{13}$$

Taking into regard that the annihilating polynomial of R is

$$R_E^2 + 2a_1 R_E + I = 0$$

we have

$$a_1^2 + b_1^2 = 1 \tag{14}$$

We can make $a_1 = -\cos\varphi$; $b_1 = \sin\varphi$

Thereby we have

$$\boxed{R_E = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}} \tag{15}$$

Therefore R_E **is a classic euclidean rotation into** E_2

4.3 Types of Lorentz transformations.

Once known the structure of orthogonal transformation on L_4 we can do a basic classification of Lorentz transformation.

4.3.1 Mixed Lorentz boost.

Outlining foregoing sections the most general Lorentz transformation (namely Lorentz boost and dim-2 euclidean rotation, described above) is:

$$(R) = \begin{pmatrix} (R_L) & (0) \\ (0) & (R_E) \end{pmatrix} \tag{16}$$

where

$$(R_L) = \begin{pmatrix} \cosh\phi & \sinh\phi \\ \sinh\phi & \cosh\phi \end{pmatrix} \tag{17}$$

and

$$R_E = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} \tag{18}$$

We call this transformation **mix Lorentz boost**. It is made up a Lorentz boost and a dim-2 eucledian rotation.

4.3.2 Pure Lorentz boost.

Herein we have

$$(R) = \begin{pmatrix} (R_L) & (0) \\ (0) & (I_E) \end{pmatrix} \tag{19}$$

where

$$(R_L) = \begin{pmatrix} \cosh\phi & \sinh\phi \\ \sinh\phi & \cosh\phi \end{pmatrix} \tag{20}$$

Herein we are dealing with the case in which $\cos\varphi = 0$. Therefore R_E becomes I_E. See footnote 6.

$$I_E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{21}$$

We call this transformation **pure Lorentz boost**. It is the classic relativistic transformation studied in basics books of relativity. The Special Relativity Theory is constructed on the basis of the interpretation of implications that pure **Lorentz boost** has in physics.For example it involves time dilatation, space contraction, and so on and so forth.

In the 2-dim euclidean space (let be the base $(\overrightarrow{\mathbf{Y}}, \overrightarrow{\mathbf{Z}})$; herein $Y, Z \in E_2$ and make up an orthonormal base) we have:

$$I_E(\overrightarrow{\mathbf{Y}}) = \overrightarrow{\mathbf{Y}}$$

$$I_E(\overrightarrow{\mathbf{Z}}) = \overrightarrow{\mathbf{Z}}$$

Then it is inferred that we are dealing with a rotation around $(\overrightarrow{\mathbf{Y}}, \overrightarrow{\mathbf{Z}})$, that is around the plane E_2.

5 Conclusions

In the I Part "*Insights into Relativity Theory. Part I. Critical approach . Basic principles and start points*" it was proved the invariance of the skew-adjoint characteristic of the electromagnetic field -or rather of the endomorphism associated to electromagnetic tensor- under orthogonal transformations , namely Lorentz transformation. The electromagnetic interactions and specially light signals must be preserved by the mentioned orthogonal transformations.

We infer the equations of the annihilating polynomial for the orthogonal endomorphism. Agree on [6] the 4 dim lorentzian vectorial space is decomposed in a 2-dim lorentzian space-time and a 2 dim euclidean space.

In this paper we have derived the Lorentz transformations (Lorentz boost and orthogonal euclidean 2 dim rotations) on the basis of the analysis and study of the cited orthogonal endomorphism throwing it in a new light.

ANNEXES

A Some propositions about lorentzian vectorial spaces geometry.

In this article some propositions and definitions necessaries to deal with vectorial spacetime, are shown. Most of these propositions are shown without proving since it is not the subject of this article.

As far as I know the most thorough study about lorentzian vectorial space is in [1], [3], and [14] .

L_n stands for a lorentzian space of signature (-1,1,....1). In the spacetime L_4 we are limited to dimension 4, signature (-1,1,1,1).

E_i $i = 1, 2, 3$ stands for an euclidian subspace of L_4.

I_1 stands for a null straight line.

I_k $k = 1, 2, 3, 4$ stands for a null subspace or space.

A.1 Definitions.

Spacelike
It is easily checked that a subspace generated by **orthogonal** spacelike vectors , is an euclidian subspace. This euclidian subspace is named *spacelike subspace.*

All his vectors are spacelike.

A subspace spacelike is euclidian.

Causal subspace

It contains timelike, spacelike and null vectors.

Null subspace

It is a subspace formed of a null vector and a subspace orthogonal to it. Vectors of this orthogonal space are *spacelike.*

If the subspace orthogonal to the null vector is a 3D spacelike , then this is constituted by null vector and three spacelike vectors ortogonal to it. This space is called *properly null space* I_4

A *properly null space* is solely formed by one null vector and also a subspace generated by three spacelike vectors orthogonal to the null vector. It does not contains timelike vectors.

Vectors orthogonal to a causal subspace, are spacelike vectors. Thereby a subspace orthogonal to a null vector is spacelike.

In a null subspace, a reference frame formed by spacelike, is moving at light speed.

A.2 Orthogonality relations.

Two timelike vectors can not be orthogonal.

If two null vectors are orthogonal then they are proportional. They are orthogonal to themselves.

A vector orthogonal to a timelike vector is spacelike vector. A vector orthogonal to a null vector, is spacelike vector or null vector.

A vector orthogonal to a spacelike vector is spacelike or timelike or null vector.

B Concepts about referential frame at rest respect an observer.

B.1 Definition of reference frame at rest respect an observer.

We define a reference frame at rest related to an observer, as reference frame such that the matrix $(\mathbf{G_r})$ of the metric tensor covariant or contravariant components in this reference frame is:

$$(\mathbf{G_r}) = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

Thereby the vectorial base of the reference frame at rest respect an observer must be orthonormal.

B.2 Passage from the pseudo-orthonormal base to an orthonormal base at rest.

That is from the pseudo-orthonormal base $(\overrightarrow{\mathbf{p}}, \overrightarrow{\mathbf{q}}, \overrightarrow{\mathbf{Y}}, \overrightarrow{\mathbf{Z}})$ to the orthonormal base $(\overrightarrow{\mathbf{U}}, \overrightarrow{\mathbf{X}}, \overrightarrow{\mathbf{Y}}, \overrightarrow{\mathbf{Z}})$

As we saw earlier in a pseudo-orthonormal base we have:

$$\overrightarrow{\mathbf{p}}^2 = \overrightarrow{\mathbf{q}}^2 = 0; \overrightarrow{\mathbf{p}}.\overrightarrow{\mathbf{q}} = w; \overrightarrow{\mathbf{Y}}.\overrightarrow{\mathbf{Z}} = \overrightarrow{\mathbf{p}}.\overrightarrow{\mathbf{Y}} = \overrightarrow{\mathbf{p}}.\overrightarrow{\mathbf{Z}} = \overrightarrow{\mathbf{q}}.\overrightarrow{\mathbf{Y}} = \overrightarrow{\mathbf{q}}.\overrightarrow{\mathbf{Z}} = 0$$

$$\overrightarrow{\mathbf{Y}}^2 = \overrightarrow{\mathbf{Z}}^2 = 1$$

In the orthonormal basis $(\overrightarrow{\mathbf{U}}, \overrightarrow{\mathbf{X}}, \overrightarrow{\mathbf{Y}}, \overrightarrow{\mathbf{Z}})$ we have

$$\overrightarrow{\mathbf{U}}^2 = -1; \overrightarrow{\mathbf{X}}^2 = \overrightarrow{\mathbf{Y}}^2 = \overrightarrow{\mathbf{Z}}^2 = 1$$

$$\overrightarrow{\mathbf{U}}.\overrightarrow{\mathbf{X}} = \overrightarrow{\mathbf{U}}.\overrightarrow{\mathbf{Y}} = \overrightarrow{\mathbf{U}}.\overrightarrow{\mathbf{Z}} = \overrightarrow{\mathbf{X}}.\overrightarrow{\mathbf{Y}} = \overrightarrow{\mathbf{X}}.\overrightarrow{\mathbf{Z}} = \overrightarrow{\mathbf{Y}}.\overrightarrow{\mathbf{Z}} = 0$$

The passage equations are:

$$\overrightarrow{\mathbf{p}} = a_p\overrightarrow{\mathbf{X}} + b_p\overrightarrow{\mathbf{U}}$$

$$\overrightarrow{\mathbf{q}} = a_q\overrightarrow{\mathbf{X}} + b_q\overrightarrow{\mathbf{U}}$$

$\overrightarrow{\mathbf{Y}}$ and $\overrightarrow{\mathbf{Z}}$ remain the same.

It is easily checked that:

$$a_p = \varepsilon b_p = a$$

$$a_q = \eta b_q = b$$

It must be $\eta = -\varepsilon$. Then the transition equations become:

$$\overrightarrow{\mathbf{p}} = a(\varepsilon\overrightarrow{\mathbf{U}} + \overrightarrow{\mathbf{X}})$$

$$\overrightarrow{\mathbf{q}} = b(-\varepsilon\overrightarrow{\mathbf{U}} + \overrightarrow{\mathbf{X}})$$

Here we have $\overrightarrow{\mathbf{p}}.\overrightarrow{\mathbf{q}} = 2ab = w$.

To keep the orientation toward the future must be $a > 0$, $b < 0$, $\varepsilon = +1$, thereby $w = 2ab < 0$

Then

$$\overrightarrow{\mathbf{p}} = a(\overrightarrow{\mathbf{U}} + \overrightarrow{\mathbf{X}})$$
$$\overrightarrow{\mathbf{q}} = b(-\overrightarrow{\mathbf{U}} + \overrightarrow{\mathbf{X}})$$

C Notations, symbols and terminology.

Vectors are symbolized with over right arrow.

Tensors stand for bold uppercase letters.

Endomorphisms stand for normal uppercase letters.

The matrix of components of an endomorphism, tensor, etc.. is shown closing inside parenthesis the symbol of this endomorphism, tensor, etc.. . For example $(\mathbf{T})$ stands for the matrix of components of $\mathbf{T}$. $(\mathbf{g}_{\alpha\beta})$ is a matrix whose elements are $\mathbf{g}_{\alpha\beta}$.

However for convenience we manage without parenthesis when specified.

The two vectors scalar product $\overrightarrow{\mathbf{x}}$ and $\overrightarrow{\mathbf{y}}$ is symbolized by $\mathbf{G}(\overrightarrow{\mathbf{x}}, \overrightarrow{\mathbf{y}})$ where $\mathbf{G}$ is the metric tensor. Also is symbolized by $\overrightarrow{\mathbf{x}}.\overrightarrow{\mathbf{y}}$.

Subscripts are symbolized by lower case greek or latin letters, saving λ and μ that are used to denote invariants.

Usually E_2 symbolize a 2 dim euclidean space, L_2 a 2 dim vectorial lorentzian space and L_n a n-dim vectorial lorentzian space. Meanwhile we do not know if the space is lorentzian L_n or euclidean E_n we symbolize these spaces with symbol $\mathbb{L}_n$. In general if it is not established if the space is lorentzian or eucledian we will use the blackboard bold letter [8] to represent the space.

$T^\sharp$ is de G-adjoint endomorphism of T. T^t is de transposed endomorphism of T.

In regard to the called *endomorphism associated to a tensor* it is necessary to make clear that the components of the mentioned endomorphism are those of the mixed components of the tensor.

[8] for example L in blackboard bold letter is $\mathbb{L}$.

As far I know there are not a bibliography fitted for this paper. But next I show some suitable publications. However, to a large extent, I have developed my work on the basis of this bibliography.

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