Addendum to ”Structures of the Skew-adjoint Endomorphisms and Some Peculiarities of Electromagnetic Field”

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Addendum to "Structures of the Skew-adjoint Endomorphisms and Some Peculiarities of Electromagnetic Field". *

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Abstract

In above mentioned paper some items were up in the air. We intend develop them here. These items were not basic and did not affect to above mentioned paper.

As we will describe after, the items we show in this paper are to fix the w value and the α and β values.

*Together with the analysis and study of skew-adjoint endomorphisms on the basis of invariant subspaces, we make a presentation of electromagnetic field structure tensor in the framework of the Relativity.

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1 Introduction

In our paper *"Structures of the Skew-adjoint Endomorphisms and Some Peculiarities of Electromagnetic Field"* the metric tensor in his covariant form, is

$$(G) = \begin{pmatrix} 0 & w & & \\ w & 0 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

using the pseudo orthogonal reference frame $(\overrightarrow{\mathbf{p}}, \overrightarrow{\mathbf{q}}, \overrightarrow{\mathbf{Y}}, \overrightarrow{\mathbf{Z}})$ (see our paper *"Structures of the Skew-adjoint Endomorphisms and Some Peculiarities of Electromagnetic Field"* Annex B.2).

It is verified

$$\overrightarrow{\mathbf{p}}^2 = \overrightarrow{\mathbf{q}}^2 = 0; \overrightarrow{\mathbf{p}}.\overrightarrow{\mathbf{q}} = w; \overrightarrow{\mathbf{Y}}.\overrightarrow{\mathbf{Z}} = \overrightarrow{\mathbf{p}}.\overrightarrow{\mathbf{Y}} = \overrightarrow{\mathbf{p}}.\overrightarrow{\mathbf{Z}} = \overrightarrow{\mathbf{q}}.\overrightarrow{\mathbf{Y}} = \overrightarrow{\mathbf{q}}.\overrightarrow{\mathbf{Z}} = 0$$

$$\overrightarrow{\mathbf{Y}}^2 = \overrightarrow{\mathbf{Z}}^2 = 1$$

In the orthogonal basis $(\overrightarrow{\mathbf{U}}, \overrightarrow{\mathbf{X}}, \overrightarrow{\mathbf{Y}}, \overrightarrow{\mathbf{Z}})$, the metric tensor is

$$(g) = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

and we have

$$\overrightarrow{\mathbf{U}}^2 = -1; \overrightarrow{\mathbf{X}}^2 = \overrightarrow{\mathbf{Y}}^2 = \overrightarrow{\mathbf{Z}}^2 = 1$$

$$\overrightarrow{\mathbf{U}}.\overrightarrow{\mathbf{X}} = \overrightarrow{\mathbf{U}}.\overrightarrow{\mathbf{Y}} = \overrightarrow{\mathbf{U}}.\overrightarrow{\mathbf{Z}} = \overrightarrow{\mathbf{X}}.\overrightarrow{\mathbf{Y}} = \overrightarrow{\mathbf{X}}.\overrightarrow{\mathbf{Z}} = \overrightarrow{\mathbf{Y}}.\overrightarrow{\mathbf{Z}} = 0$$

The passage equations between the pseudo orthogonal reference frame and the orthogonal reference frame, are:

$$\overrightarrow{\mathbf{p}} = a_p\overrightarrow{\mathbf{X}} + b_p\overrightarrow{\mathbf{U}}$$

$$\overrightarrow{\mathbf{q}} = a_q\overrightarrow{\mathbf{X}} + b_q\overrightarrow{\mathbf{U}}$$

$\overrightarrow{\mathbf{Y}}$ and $\overrightarrow{\mathbf{Z}}$ remain the same.

It is easily checked that:

$$a_p = \varepsilon b_p = a$$

$$a_q = \eta b_q = b$$

It must be $\eta = -\varepsilon$. Then the transition equations become:

$$\mathbf{p} = a(\varepsilon\overrightarrow{\mathbf{U}} + \overrightarrow{\mathbf{X}})$$

$$\mathbf{q} = b(-\varepsilon\overrightarrow{\mathbf{U}} + \overrightarrow{\mathbf{X}})$$

Here we have $\overrightarrow{\mathbf{p}}.\overrightarrow{\mathbf{q}} = 2ab = w$.

To keep the orientation toward the future must be $a > 0$, $b < 0$. $\varepsilon = +1$, thereby $w = 2ab < 0$

For this reason we make $a = \alpha$, $b = -\beta$, being $\alpha > 0$, $\beta > 0$.

Exceptionally, only in this section and in the next, we use α and β, not as subindex but as parameters.

Then

$$\mathbf{p} = \alpha(\overrightarrow{\mathbf{U}} + \overrightarrow{\mathbf{X}})$$

$$\mathbf{q} = \beta(\overrightarrow{\mathbf{U}} - \overrightarrow{\mathbf{X}})$$

and $w = -2\alpha\beta$

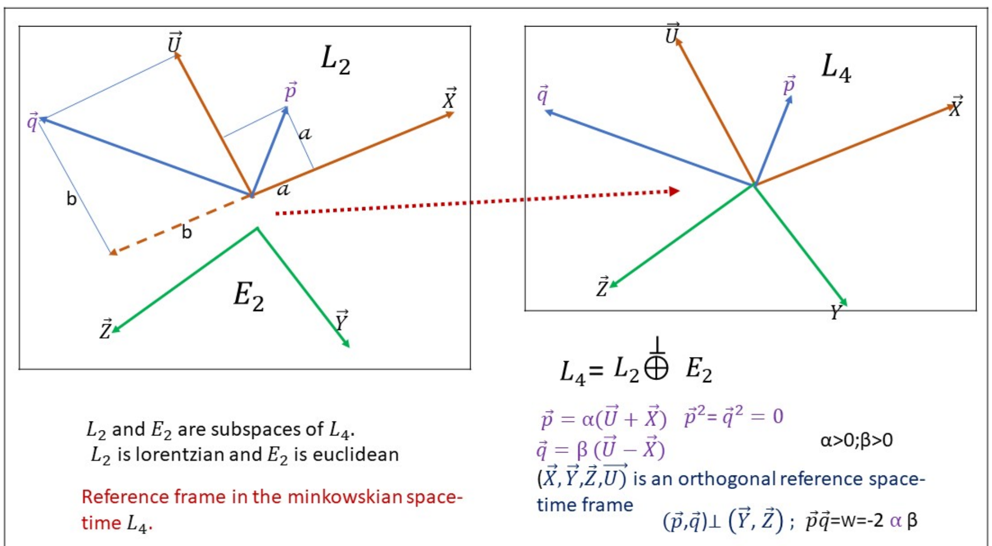

Fig1

2 Fixing the w value

The passage of (G) to (g) is

$$(g) = B^t (G) B$$

where B depicts the above mentioned passage.

$$(B) = \begin{pmatrix} \alpha & \alpha & & \\ \beta & -\beta & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

$$(g) = \begin{pmatrix} \alpha & \beta & & \\ \alpha & -\beta & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} 0 & w & & \\ w & 0 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} \alpha & \alpha & & \\ \beta & -\beta & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -w^2 & & & \\ & w^2 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

It is clear that

$$w^2 = 1 \text{ , thereby } w = -1$$

Let us now fix the α and β values.

In the orthogonal basis $R_g \equiv (\overrightarrow{\mathbf{U}}, \overrightarrow{\mathbf{X}}, \overrightarrow{\mathbf{Y}}, \overrightarrow{\mathbf{Z}})$, null vectors $\overrightarrow{\mathbf{p}}$ and $\overrightarrow{\mathbf{q}}$ have the next components:

$$\overrightarrow{\mathbf{p}} = (\alpha, \alpha, 0, 0)$$

$$\overrightarrow{\mathbf{q}} = (\beta, -\beta, 0, 0)$$

In the pseudo orthogonal basis $R_G \equiv (\overrightarrow{\mathbf{p}}, \overrightarrow{\mathbf{q}}, \overrightarrow{\mathbf{Y}}, \overrightarrow{\mathbf{Z}})$, null vectors $\overrightarrow{\mathbf{p}}$ and $\overrightarrow{\mathbf{q}}$ have the next components:

$$\overrightarrow{\mathbf{p}} = (1, 0, 0, 0)$$

$$\overrightarrow{\mathbf{q}} = (0, 1, 0, 0)$$

Here the equations of passage from R_g to R_G are

$$(\overrightarrow{\mathbf{U}}, \overrightarrow{\mathbf{X}}, \overrightarrow{\mathbf{Y}}, \overrightarrow{\mathbf{Z}}) \overset{\mathbf{B^{-1}}}{\Rightarrow} (\overrightarrow{\mathbf{p}}, \overrightarrow{\mathbf{q}}, \overrightarrow{\mathbf{Y}}, \overrightarrow{\mathbf{Z}})$$

where

$$(\mathbf{B^{-1}}) = \begin{pmatrix} -\beta & -\alpha & & \\ -\beta & \alpha & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \cdot \frac{1}{w}$$

being $w = -2\alpha\beta$

$$\frac{1}{-2\alpha\beta} \begin{pmatrix} -\beta & -\alpha & & \\ -\beta & \alpha & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \alpha \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

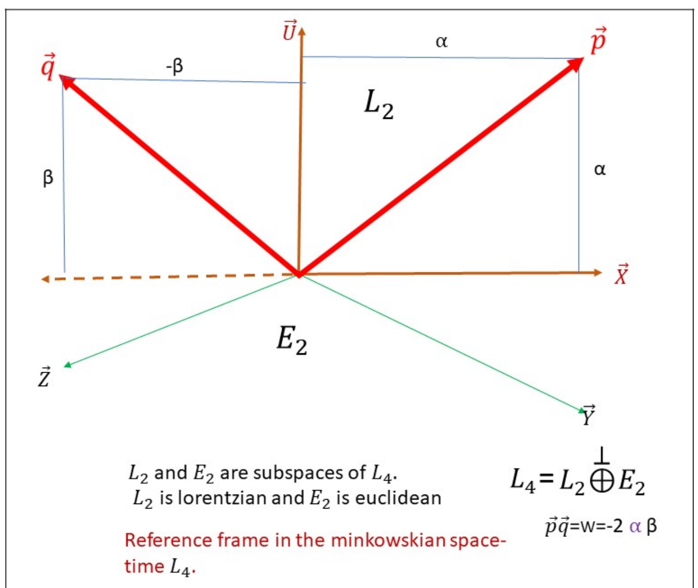

Fig 2

Therefore

$$\alpha = \beta \ ; \ 2\alpha^2 = 1 \ ; \ \alpha = \frac{\sqrt{2}}{2}$$

Thereby

$$\boxed{\mathbf{p} = \frac{\sqrt{2}}{2}(\overrightarrow{\mathbf{U}} + \overrightarrow{\mathbf{X}})}$$

$$\boxed{\mathbf{q} = \frac{\sqrt{2}}{2}(\overrightarrow{\mathbf{U}} - \overrightarrow{\mathbf{X}})}$$

3 Representation of the tensor of the electromagnetic field

In accordance with the paper *"Structures of the Skew-adjoint Endomorphisms and Some Peculiarities of Electromagnetic Field"* w appears only in the reference frame $(\overrightarrow{\mathbf{p}}, \overrightarrow{\mathbf{q}}, \overrightarrow{\mathbf{Y}}, \overrightarrow{\mathbf{Z}})$ in the covariant and contravariant forms.

In contravariant form, the expression of electromagnetic field is

$$\mathbf{F} = \lambda w \overrightarrow{\mathbf{p}} \wedge \overrightarrow{\mathbf{q}} + \mu \overrightarrow{\mathbf{Y}} \wedge \overrightarrow{\mathbf{Z}}$$

As $w = -1$ we have

$$\boxed{\mathbf{F} = -\lambda \overrightarrow{\mathbf{p}} \wedge \overrightarrow{\mathbf{q}} + \mu \overrightarrow{\mathbf{Y}} \wedge \overrightarrow{\mathbf{Z}}}$$

Here, $\overrightarrow{\mathbf{p}}, \overrightarrow{\mathbf{q}}, \overrightarrow{\mathbf{Y}}, \overrightarrow{\mathbf{Z}}$, are contravariant vectors.

In the above mentioned paper, the electromagnetic field, in covariant form, appears to be

$$\mathbf{F} = -\tfrac{1}{w}\lambda \overrightarrow{\mathbf{p}} \wedge \overrightarrow{\mathbf{q}} + \mu \overrightarrow{\mathbf{Y}} \wedge \overrightarrow{\mathbf{Z}}$$

In this expression, $\overrightarrow{\mathbf{p}}, \overrightarrow{\mathbf{q}}, \overrightarrow{\mathbf{Y}}, \overrightarrow{\mathbf{Z}}$ are covariant vectors.

As we have $w = -1$, the previous expression turns into

$$\boxed{\mathbf{F} = \lambda \overrightarrow{\mathbf{p}} \wedge \overrightarrow{\mathbf{q}} + \mu \overrightarrow{\mathbf{Y}} \wedge \overrightarrow{\mathbf{Z}}}$$

In the reference frame $(\overrightarrow{\mathbf{U}}, \overrightarrow{\mathbf{X}}, \overrightarrow{\mathbf{Y}}, \overrightarrow{\mathbf{Z}})$, in the covariant form, we have

$$\mathbf{F} = \lambda \overrightarrow{\mathbf{U}} \wedge \overrightarrow{\mathbf{X}} + \mu \overrightarrow{\mathbf{Y}} \wedge \overrightarrow{\mathbf{Z}}$$

As we can see, in this expression $\mathbf{F}$ does not depend on w.

4 Dual tensors of F

The dual of electromagnetic tensor of $\mathbf{F}_{\gamma\delta}$ is

$$\mathbf{F}^{*}_{\gamma\delta} = \varepsilon_{\gamma\delta\rho\sigma}\mathbf{F}^{\rho\sigma}$$

$$\gamma < \delta < \rho < \sigma$$

As we have seen before, it is clear that

$$\mathbf{F} = \lambda\overrightarrow{\mathbf{U}} \wedge \overrightarrow{\mathbf{X}} + \mu\overrightarrow{\mathbf{Y}} \wedge \overrightarrow{\mathbf{Z}}$$

in covariant coordinates.

It is not hard to derive the dual tensor $\mathbf{F}^{*}$ of $\mathbf{F}$, in covariant coordinates:

$$\boxed{\mathbf{F}^{*} = -\mu\overrightarrow{\mathbf{U}} \wedge \overrightarrow{\mathbf{X}} + \lambda\overrightarrow{\mathbf{Y}} \wedge \overrightarrow{\mathbf{Z}}}$$

We arise to the fact that we get the dual of $\mathbf{F}$ swapping the invariants λ and μ and changing the sign of μ.

5 Fixing electric E field and magnetic field H.

The electric field is defined in the next way

$\overrightarrow{\mathbf{E}} = \mathbf{F}.\overrightarrow{\mathbf{U}}$

and the magnetic field

$\overrightarrow{\mathbf{H}} = \mathbf{F}^{*}.\overrightarrow{\mathbf{U}}$

As one can easily verify

$$\boxed{\overrightarrow{\mathbf{E}} = \lambda\overrightarrow{\mathbf{X}}}$$

in covariant components

and

$$\boxed{\overrightarrow{\mathbf{H}} = -\mu\overrightarrow{\mathbf{X}}}$$

in covariant components as well.

The electric vectorial field $\overrightarrow{\mathbf{E}}$, and magnetic vectorial field $\overrightarrow{\mathbf{H}}$, in this section, are determined in the reference frame shown in *figure 1*. Respect other referencial frames, $\overrightarrow{\mathbf{E}}$ and $\overrightarrow{\mathbf{H}}$ could be in different position in the spacelike.

References

[1] R. K. Sachs ;General Relativity for mathematicians (Springer Verlag ; New York 1977); Chapters 1,2,8 y 9

[2] (gigda.ugr.es/sanchezm/data/uploads /Apuntes1EspVectIndefyLorentz.pdf); Chapter 1

[3] H. Stephani ;General Relativity (Cambridge University Press 1982). Special properties of source free fields; pag 72

[4] B. Coll;Concepts for a theory of the electromagnetic field (2003) gr-qc/0302056v1; a full explanation of the formulation of electromagnetic field

[5] Marsall C. Pease III; Methods of Matrix Algebra; Academic Press 1965; pag 196 and next; spectral theorem.

[6] F.R. Gantmacher; The theory of matrices; Chelsea 1960; chapters V and VII

[7] Electrovacuum solutions; wikipedia; references Stephani and others ; Exact solutuion of Einstein equations