# Structures of the Skew-adjoint Endomorphisms and Some Peculiarities of Electromagnetic Field .

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# Structures of the Skew-adjoint Endomorphisms and Some Peculiarities of Electromagnetic Field.
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November 7, 2014

**Abstract**

We throw a new light on skew-adjoint endomorphisms and electromagnetic field. They are presented in the framework of lorentzian space-time of dimension 4, $L_4$ with signature (-1,1,1,1). Kinds of endomorphisms and associated tensors are inferred on $L_4$. The inferred structures are applied to electromagnetic fields.

We find a more general tensorial representation of electromagnetic tensor.

In general this analysis is conformed to two types of electromagnetic tensors we know.

*Together with the analysis and study of skew-adjoint endomorphisms on the basis of invariant subspaces, we make a presentation of electromagnetic field structure tensor in the framework of General Relativity.

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## 1 Introduction

The purpose of this article is to bring out some new insights about topics that come into view in the field of skew-adjoint endomorphisms in connection with the electromagnetic field. In the analysis of the mentioned endomorphisms and in its application to electromagnetic field, interestingly we not only rediscover topics already solved, but rather we show how come up startling findings. Anyway the matters developed herein attain more general viewpoints.

The theory of skew-adjoint endomorphisms that we are developing is constructed from the analysis and study of invariants on vectorial minkowskian spaces and their subspaces on the basis of the study of annihilating polynomials, spectral theorem, etc... regarding also minimal annihilating polynomials where relevant. Afterwards this theory is applied to electromagnetic field.

The three first sections , are dealing with skew-adjoint endomorphism structure. We single out some interesting skew-adjoint endomorphisms features that verify the equation of the annihilating polynomial .

Thereafter we continue analyzing the invariants of the skew-adjoint endomorphisms . We make a comment about a question in connection with the transition to limit of the mentioned invariants.

We continue constructing the skew-adjoint tensor from skew-adjoint endomorphism defining and setting a metric beforehand.

Tracing back, into the 20th century, came out a huge number of topics about all these objets some of which we are going over here. They are amply known for everybody including students and so there are a high number of articles and books that analyze closely the mentioned objects ( see for example , [6] [8] [7] ,and so on ). It does not mean that these simple tools are now out of date to go further into some modern physical topics.

In the analysis of such structures we firstly settle in two cases already known ( pure electromagnetic field or regular case and pure radiation fields or singular cases). These cases are the most relevant in physics.

The word radiation is not suitable for this analysis. I use it resting upon the fact that it is used in the issues relating to electromagnetic field.

The radiation case ( or singular case) is not analyzed here. We are confined to the most remarkable regular cases (pure field).

I add an Annex ( Annex A) to give facilities to understand the lorentzian vectorial spaces geometry in this article. For a more detailed account see [1], [3] , [11], [2].

## 2 Notations, symbols and terminology

$\mathbf{I_{n-1}}$ stands for null cone into $L_n$.
$\mathbf{I_1}$ stands for null straight line.

Vectors are symbolized with over right arrow.
Tensors stand for bold uppercase letters.
Endomorphisms stand for normal uppercase letters.
The matrix of components of an endomorphism, tensor, etc.. is shown closing into parenthesis the symbol of this endomorphism, tensor, etc.. . For example $(\mathbf{T})$ stands for the matrix of components of $\mathbf{T}$. $(\mathbf{g}_{\alpha\beta})$ is a matrix whose elements are $\mathbf{g}_{\alpha\beta}$.

The two vectors scalar product $\overrightarrow{\mathbf{x}}$ and $\overrightarrow{\mathbf{y}}$ is symbolized by $\mathbf{G}(\overrightarrow{\mathbf{x}},\overrightarrow{\mathbf{y}})$ where $\mathbf{G}$ is the metric tensor. Also is symbolized by $\overrightarrow{\mathbf{x}}.\overrightarrow{\mathbf{y}}$.

Subindex are symbolized by lowercase greek letters, saving $\lambda$ and $\mu$ that are used to denote invariants.

If a matrix $A^2$ is decomposable, it is symbolized by $A.A$. If it is not decomposable it is symbolized simply by $A^2$

## 3 Annihilating polynomials in a lorentzian space-time.

Following a straightforward way we begin with the analysis of invariant subspaces in a lorentzian space $L_4$ constructing them on the usual basis of annihilating polynomials into $L_4$ or his subspaces. The invariant subspaces are lorentzian, euclídean or null. Thereby firstly my purpose is the analysis of annihilating polynomials regarding annihilating minimal polynomials where relevant.

### 3.1 Study of annihilating polynomial of grade 4, of a skew-adjoint endomorphism A in a minkowskian space $L_4$.

The most general case of annihilating polynomial in our context of lorentzian space $L_4$ is:

$$P(A) = A^4 + a_3A^3 + a_2A^2 + a_1A + a_0I$$

We have

$$\forall\overrightarrow{\mathbf{X}};\overrightarrow{\mathbf{X}} \in L_4; P(A)\overrightarrow{\mathbf{X}} = 0$$

Let be now $A$ a skew-adjoint endomorphism on $L_4$. It is easily checked that

$$\forall \overrightarrow{\mathbf{X}}, \overrightarrow{\mathbf{Y}} \in L_4 \quad ; \; g(\overrightarrow{\mathbf{X}}, P(A)\overrightarrow{\mathbf{Y}}) = 0 \quad g(\overrightarrow{\mathbf{Y}}, P(A)\overrightarrow{\mathbf{X}}) = 0 \tag{1}$$

Summing and subtracting equations 1 , and taking into account that $A^2$ and $A^4$ are self-adjoint endomorphisms, and that $A$ and $A^3$ are skew-adjoint endomorphisms we have

$$\forall \overrightarrow{\mathbf{X}}, \overrightarrow{\mathbf{Y}} \in L_4 \;\; ; g(\overrightarrow{\mathbf{X}}, (A^4 + a_2A^2 + a_0I)\overrightarrow{\mathbf{Y}}) = 0$$
$$; \;\; g(\overrightarrow{\mathbf{X}}, (a_3A^3 + a_1A)\overrightarrow{\mathbf{Y}}) = 0$$

One can easily verify

$$\begin{aligned} P_4(A) &\equiv A^4 + a_2A^2 + a_0I = (0) \\ P_3(A) &\equiv (A)(a_3A^2 + a_1) = (0) \end{aligned} \tag{2}$$

Only one of equations 2 is verified and the other must be identically null (his coefficients must be 0). That is only if $a_3$ and $a_1$ are 0.

Therefore in this article we only will be concerned with the first equation of 2.

So the annihilating polynomial is:

$$P_4(A) \equiv A^4 + a_2A^2 + a_0I = (0)$$

[1]

Summing up equation $P_4(A) \equiv A^4 + a_2A^2 + a_0I = (0)$, depicts annihilating polynomial on $L_4$ that concern us.

### 3.2 Factorization of Annihilating polynomial on $L_4$.

Agreeing on the foregoing sections, the goal of this article is the examination of skew-adjoint endomorphism to investigate afterward more closely electromagnetic field.

Here we are only concerned with the factorized polynomial:

$$P_4(A) \equiv (A^2 + \epsilon\lambda^2 I)(A^2 + \eta\mu^2 I) \tag{3}$$

$\epsilon = \pm 1$ ; $\eta \pm 1$

As it is easily checked

$$a_2 = \epsilon\lambda^2 + \eta\mu^2 \tag{4}$$

[1] However that does not mean we have to rule out the second case $P_3(A) \equiv (A)(a_3A^2 + a_1) = (0)$. It is worthwhile to examine it aside outside this article. However henceforth we confine only to the first equation of 2.

$$a_0 = \epsilon\eta\lambda^2\mu^2 \tag{5}$$

Further along, in other sections, we shall study the classification of annihilating polynomial $P_4(A) \equiv A^4 + a_2 A^2 + a_0 I = (0)$ on the basis of $\lambda$ and $\mu$.

### 3.3 Orthogonality relation between ker$(A^2+\epsilon\lambda^2 I)$ and ker$(A^2+\eta\mu^2 I)$

Here we shall prove:

If A is skew-adjoint endomorphism, $\lambda \neq 0$ or $\mu \neq 0$, and $\mid \lambda \mid \neq \mid \mu \mid$ then

$$L_4 \equiv ker(A^2 + \epsilon\lambda^2 I) \overset{\perp}{\oplus} ker(A^2 + \mu^2 I)$$

In fact

a)-

Let $P_\lambda(A) \equiv A^2 + \epsilon\lambda^2 I$ , $P_\mu(A) \equiv A^2 + \epsilon\mu^2 I$

$$\forall \overrightarrow{\mathbf{X}} \in ker(A^2 + \epsilon\lambda^2 I), \forall \overrightarrow{\mathbf{Y}} \in ker(A^2 + \eta\mu^2 I) \tag{6}$$

we have

$$\begin{aligned} \overrightarrow{\mathbf{Y}} A^2 \overrightarrow{\mathbf{X}} &= -\epsilon\lambda^2 \overrightarrow{\mathbf{Y}}\overrightarrow{\mathbf{X}} \\ \overrightarrow{\mathbf{X}} A^2 \overrightarrow{\mathbf{Y}} &= -\eta\mu^2 \overrightarrow{\mathbf{X}}\overrightarrow{\mathbf{Y}} \end{aligned} \tag{7}$$

We infer

$$\overrightarrow{\mathbf{X}}.\overrightarrow{\mathbf{Y}}.(\epsilon\lambda^2 - \eta\mu^2) = 0 \tag{8}$$

Taking into account the foregoing assumptions we have:

$$\forall \overrightarrow{\mathbf{X}} \in ker(A^2 + \epsilon\lambda^2 I), \forall \overrightarrow{\mathbf{Y}} \in ker(A^2 + \eta\mu^2 I) \quad \overrightarrow{\mathbf{X}}.\overrightarrow{\mathbf{Y}} = 0 \tag{9}$$

and therefore

$$ker(A^2 + \epsilon\lambda^2 I) \perp ker(A^2 + \eta\mu^2 I) \tag{10}$$

b)-

According with spectral theorem (see for example [7] ) the intersection $\overrightarrow{\mathbf{l}}$ =ker$(A^2+\epsilon\lambda^2 I)\cap$ker$(A^2+\mu\lambda^2 I)$ solely can be $\overrightarrow{\mathbf{l}}=\overrightarrow{\mathbf{0}}$. Any way we prove this intersection is empty, that is $\overrightarrow{\mathbf{l}}=\overrightarrow{\emptyset}$.

If intersection is not empty, then there is a null vector $\overrightarrow{\mathbf{l}}$ that verifies

$$\begin{aligned}(A^2+\epsilon\lambda^2 I)\overrightarrow{\mathbf{l}}=0\\(A^2+\eta\mu^2 I)\overrightarrow{\mathbf{l}}=0\end{aligned}\tag{11}$$

Subtracting equations 11 we have

$$(\epsilon\lambda^2-\eta\mu^2)\overrightarrow{\mathbf{l}}=\overrightarrow{\mathbf{0}}\tag{12}$$

taking into account our previous assumptions $\lambda\neq 0$ or $\mu\neq 0$, and $\lambda^2\neq\mu^2$ equation 12 involves $\overrightarrow{\mathbf{l}}=\overrightarrow{\mathbf{0}}$ [2]

Summing up

$$\lambda\neq 0 \quad \text{or} \quad \mu\neq 0 \quad \text{and} \quad \epsilon\lambda^2\neq\eta\mu^2\tag{13}$$

involves orthogonality between ker$(A^2+\epsilon\lambda^2 I)$ and $ker(A^2+\eta\mu^2 I)$ with zero intersection that is $L_4\equiv ker(A^2+\epsilon\lambda^2 I)\overset{\perp}{\oplus}ker(A^2+\mu^2 I)$.

**We call regular cases to the endomorphisms that verify 13.**

As complement it is worthwhile to prove : If ker$(A^2+\epsilon\lambda^2 I)\cap$ker$(A^2+\mu\lambda^2 I)\neq\emptyset$ then

$$\lambda=0 \quad \text{and} \quad \mu=0 \quad \text{or} \quad \epsilon\lambda^2=\eta\mu^2\tag{14}$$

It is not hard to prove this proposition regarding 12

In the following sections we analyze invariants subspaces of A, together with their minimal polynomials when relevant.

Notice if $\epsilon\neq\eta$ it is sufficient $\lambda\neq 0$ or $\mu\neq 0$ to verify 13 .

### 3.4 Annihilating polynomials $(A^2+\epsilon\lambda^2 I)$, $(A^2+\eta\mu^2 I)$ and their invariant subspaces structure

. We devote this section to study invariant subspaces in order to go further into their study. In this stage we are in the context 13. So we begin with the case we call *regular field or pure field* [3].

[2] In the case $\epsilon\neq\eta$ is enough $\lambda\neq 0$ or $\mu\neq 0$ to fulfil $L_4\equiv ker(A^2+\epsilon\lambda^2 I)\overset{\perp}{\oplus}ker(A^2+\mu^2 I)$

[3] *pure field or regular field* are those that fulfil the conditions itemized in previous section, that is $\lambda\neq 0$ or $\mu\neq 0$, and $\lambda^2\neq\mu^2$

Hereinbelow we analyze cases in the context of regular case. We select the case $\epsilon = 1$ and $\eta = -1$ from which we shall constitute all other cases in the framework of *pure or regular field.*

**a** )
We begin with the case of annihilating minimal polynomial of A on $L_2 \equiv \ker(A^2 + \epsilon\lambda^2 I)$, $\epsilon = -1$ ; $\lambda \neq 0$. One easily verifies:

$$A^2 - \lambda^2 I = (A - \sigma\lambda I)(A + \sigma\lambda I) = (0)$$

$\sigma = \pm 1$

Then we have on $L_2$

$$(A - \sigma\lambda I)\overrightarrow{\mathbf{p}} = \overrightarrow{\mathbf{0}}, \overrightarrow{\mathbf{p}} \in ker(A - \sigma\lambda I)$$
$$(A + \sigma\lambda I)\overrightarrow{\mathbf{q}} = \overrightarrow{\mathbf{0}}, \overrightarrow{\mathbf{q}} \in ker(A + \sigma\lambda I)$$

namely

$$\begin{aligned} A\overrightarrow{\mathbf{p}} &= \sigma\lambda\overrightarrow{\mathbf{p}} \\ A\overrightarrow{\mathbf{q}} &= -\sigma\lambda\overrightarrow{\mathbf{q}} \end{aligned} \tag{15}$$

Let $L_2 \equiv ker(A^2 - \lambda^2 I)$ :
Agreeing on 13 it is $\lambda \neq 0$
Using a classical language we tell that A has two eigenvectors $\overrightarrow{\mathbf{X}}$ and $\overrightarrow{\mathbf{Y}}$ with eigenvalues respectively $+\sigma\lambda$ and $-\sigma\lambda$.
Furthermore taking into regard that A is skew-adjoint we have

$$\overrightarrow{\mathbf{p}}A\overrightarrow{\mathbf{p}} = 0 = \sigma\lambda\overrightarrow{\mathbf{p}}^2$$
$$\overrightarrow{\mathbf{q}}A\overrightarrow{\mathbf{q}} = 0 = -\sigma\lambda\overrightarrow{\mathbf{q}}^2; \sigma = \pm 1$$

Since $\lambda \neq 0$ we have $\overrightarrow{\mathbf{p}}^2 = \overrightarrow{\mathbf{q}}^2 = 0$, thereby $\overrightarrow{\mathbf{p}}$ and $\overrightarrow{\mathbf{q}}$ are two null eigenvectors. Thereby $L_2$ is lorentzian.

In short:
In $L_2$ **there are two null eigenvectors** $\overrightarrow{\mathbf{p}}$ and $\overrightarrow{\mathbf{q}}$ **with eigenvalues** $+\sigma\lambda$ **and** $-\sigma\lambda$, respectively ; $\sigma = \pm 1$.
$L_2$ is a lorentzian subspace because has two null vectors. His orthogonal complementary is $ker(A^2 + \eta\mu^2 I)$ in accordance with 3.3. Therefore the forth dimensional lorentzian space that concern us is :

$$L_4 = L_2 \overset{\perp}{\oplus} ker(A^2 + \eta\mu^2 I)$$

where $L_2 \equiv Ker(A - \lambda I)(A + \lambda I)$.

**b)-** Let us analyze now the structure of $E_2 \equiv ker(A^2 + \eta\mu^2 I)$ for $\eta = +1$. Here $\mu \neq 0$ agreeing on previous assumption.

One easily verifies
$\forall \overrightarrow{\mathbf{Y}} \in ker(A^2 + \mu^2 I)$ , $\exists \overrightarrow{\mathbf{Z}}$;

$$A\overrightarrow{\mathbf{Y}} = \nu\mu\overrightarrow{\mathbf{Z}}$$

$$A\overrightarrow{\mathbf{Z}} = -\nu\mu\overrightarrow{\mathbf{Y}}, \quad ;\nu = \pm 1$$

Taking into account that A is skew-adjoint, we have:

$$\overrightarrow{\mathbf{Y}}A\overrightarrow{\mathbf{Y}} = 0 = \mu\overrightarrow{\mathbf{Y}}.\overrightarrow{\mathbf{Z}}$$

$$\overrightarrow{\mathbf{Z}}A\overrightarrow{\mathbf{Z}} = 0 = -\mu\overrightarrow{\mathbf{Y}}.\overrightarrow{\mathbf{Z}}$$

$$\overrightarrow{\mathbf{Y}}A\overrightarrow{\mathbf{Z}} + \overrightarrow{\mathbf{Z}}A\overrightarrow{\mathbf{Y}} = 0 = \mu\overrightarrow{\mathbf{Z}}^2 - \mu\overrightarrow{\mathbf{Y}}^2$$

Then, as $\mu \neq 0$, we have $(\overrightarrow{\mathbf{Y}}^2 - \overrightarrow{\mathbf{Z}}^2) = 0$ namely $\overrightarrow{\mathbf{Y}}^2 = \overrightarrow{\mathbf{Z}}^2$
It is not hard to see that $\overrightarrow{\mathbf{Z}} \in E_2$
Thereby both $\overrightarrow{\mathbf{Y}}, \overrightarrow{\mathbf{Z}} \in E_2$ verify

$$\begin{aligned} A\overrightarrow{\mathbf{Y}} &= \nu\mu\overrightarrow{\mathbf{Z}} \\ A\overrightarrow{\mathbf{Z}} &= -\nu\mu\overrightarrow{\mathbf{Y}} \\ \overrightarrow{\mathbf{Y}}.\overrightarrow{\mathbf{Z}} = 0 \ \text{ and } \ \overrightarrow{\mathbf{Y}}^2 &= \overrightarrow{\mathbf{Z}}^2 \end{aligned} \tag{16}$$

$\nu = \pm 1$

Here we have two possibilities:
1)—————
$\overrightarrow{\mathbf{Y}}$ and $\overrightarrow{\mathbf{Z}}$ are null vectors. Herein it is verified $\overrightarrow{\mathbf{Y}}^2 = \overrightarrow{\mathbf{Z}}^2 = 0$, and $\overrightarrow{\mathbf{Y}}.\overrightarrow{\mathbf{Z}} = 0$. Thereby $\overrightarrow{\mathbf{Y}} = a\overrightarrow{\mathbf{Z}}$ that is $\overrightarrow{\mathbf{Y}}$ and $\overrightarrow{\mathbf{Z}}$ are proportional.

Agreeing on 16 we infer $\overrightarrow{\mathbf{Y}} = a\overrightarrow{\mathbf{Z}} = \overrightarrow{\mathbf{0}}$, so we rule out that $\overrightarrow{\mathbf{X}}$ and $\overrightarrow{\mathbf{Y}}$ would be null vectors.
2)—————-
$\overrightarrow{\mathbf{Y}}$ and $\overrightarrow{\mathbf{Z}}$ are nonull vectors.

They both can be timelike, or spacelike. If they are timelike they must be orthogonal between them in contradiction with assumptions ( proposition settled in Annex A). Thereby we rule out $\overrightarrow{\mathbf{Y}}$ and $\overrightarrow{\mathbf{Z}}$ timelike. **So $\mathbf{E_2}$ is spacelike**.

Summing up

$L_2 \equiv ker(A^2 - \lambda^2 I)$ ; $E_2 \equiv ker(A^2 + \mu^2 I)$ ; $L_4 = L_2 \overset{\perp}{\oplus} E_2$

Henceforth we shall use these notations $L_2$ and $E_2$ as $ker(A^2 - \lambda^2 I)$ and $ker(A^2 + \mu^2 I)$

This case is called pure field or regular case.

**It is not hard to prove that all other regular cases are composed of cases 1) and 2) or they are incongruous in our context.**

# 4 Invariants of skew-adjoints endomorphisms

.

Continuing with assumptions $\lambda \neq 0$ or $\mu \neq 0$, and $\lambda^2 \neq \mu^2$ as in foregoing section, annihilating minimal polynomials coefficients [4], in $L_4$, are the invariants of the endomorphism. Functions that depend only on them are invariants too. In a minimal polynomial $P(A) = A^4 + a_2 A^2 + a_0 I$, $a_2$ and $a_0$ and also $\lambda$ and $\mu$ are invariants of the skew-adjoint endomorphism A.

## 4.1 Invariants $\lambda$ and $\mu$

.

The minimal polynomial on $L_4$ that concerns us is

$$P(A) = A^4 + a_2 A^2 + a_0 I \equiv (A^2 + \epsilon\lambda^2 I)(A^2 + \eta\mu^2 I)$$

$\lambda$ and $\mu$ are functions of coefficients ( therefore invariants ) in $P(A)$.

In our analysis $\epsilon = -1$ and $\eta = 1$, agreeing on previous sections.

Then we have the invariants

$$\begin{aligned} I_1 &= \mu^2 - \lambda^2 = a_2 \\ I_2 &= -\mu^2\lambda^2 = a_0 \end{aligned} \tag{17}$$

Further along, it will be single out that the invariants $\lambda$ and $\mu$ ( or $I_1$ and $I_2$) play a relevant role in electromagnetic field.

Notice a family of similar skew-adjoint endomorphisms have the same invariants, since they have the same minimum polynomial and therefore the same coefficients.

# 5 Tensorial representation of skew-adjoint endomorphisms

In this section we shall illustrate the tensorial applications going into details, that is developing the tensor components in the two reference frames we show in the next.

[4] in this case we are dealing with minimal polinomials

In an endomorphism A *associated* to a tensor **A**, the components of endomorphism A are the mix components of a two-order tensor **A** with the same basis in they both ( A and **A**).

That is we have $(A_{\alpha\beta}) = (\mathbf{A}_{\alpha}{}^{\beta})$.

As a matter of fact, the tensorial mix components $\mathbf{A}_{\alpha}{}^{\beta}$ and the components of the associated endomorphism are the same (in connection with the same base).

The tensorial representations involves a metric tensor **G**, in which covariants components are $g_{\alpha\beta}$ and in which contravariants components are $g^{\alpha\beta}$.

Therefore we have

$$(\mathbf{A}^{\alpha\beta}) = (\mathbf{g}^{\alpha\lambda}\mathbf{A}_{\lambda}{}^{\beta})$$

### 5.1 Tensorial representation applied to regular case (pure field)

We are working on $L_2 \overset{\perp}{\oplus} E_2$

$(\overrightarrow{\mathbf{p}}, \overrightarrow{\mathbf{q}})$ is a base on $L_2$, and $(\overrightarrow{\mathbf{Y}}, \overrightarrow{\mathbf{Z}})$ is an ortonormal base on $E_2$.

In this article I develop the components of tensor **A** in two reference frames: the mentioned $(\overrightarrow{\mathbf{p}}, \overrightarrow{\mathbf{q}}, \overrightarrow{\mathbf{Y}}, \overrightarrow{\mathbf{Z}})$ and the $(\overrightarrow{\mathbf{U}}, \overrightarrow{\mathbf{X}}, \overrightarrow{\mathbf{Y}}, \overrightarrow{\mathbf{Z}})$ where

$$\overrightarrow{\mathbf{p}} = a(\overrightarrow{\mathbf{X}} + \varepsilon\overrightarrow{\mathbf{U}})$$

$$\overrightarrow{\mathbf{q}} = b(\overrightarrow{\mathbf{X}} - \varepsilon\overrightarrow{\mathbf{U}}); \varepsilon = \pm 1$$

( see Annex B)

### 5.2 Reference frame $(\vec{\mathbf{p}}, \vec{\mathbf{q}}, \vec{\mathbf{Y}}, \vec{\mathbf{Z}})$

In $L_2$ we rest upon the existence of the scalar product $w = \overrightarrow{\mathbf{p}}.\overrightarrow{\mathbf{q}}$ to define the metric on $L_2$. This is in covariant components :

$$(g_{\alpha\beta}) = \begin{pmatrix} 0 & w \\ w & 0 \end{pmatrix}$$

; $\alpha, \beta = 0, 1$

and in contravariants components

$$(g^{\alpha\beta}) = \begin{pmatrix} 0 & \frac{1}{w} \\ \frac{1}{w} & 0 \end{pmatrix}$$

; $\alpha, \beta = 0, 1$

In $E_2$ the metric is euclidian, so

$$(g_{\alpha\beta}) = (g^{\alpha\beta}) = (g_\lambda{}^\beta) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

; $\alpha, \beta = 2, 3$

In the reference frame $(\vec{\mathbf{p}}, \vec{\mathbf{q}}, \vec{\mathbf{Y}}, \vec{\mathbf{Z}})$ the metric in $L_4$ in covariant components is

$$(G_{\alpha\beta}) = \begin{pmatrix} 0 & w & 0 & 0 \\ w & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} ; \alpha, \beta = 0, 3$$

and in contravariant components

$$(G^{\alpha\beta}) = \begin{pmatrix} 0 & \frac{1}{w} & 0 & 0 \\ \frac{1}{w} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} ; \alpha, \beta = 0, 3$$

Agreeing on foregoing propositions we have $\overrightarrow{\mathbf{p}}, \overrightarrow{\mathbf{q}}, \overrightarrow{\mathbf{Y}}, \overrightarrow{\mathbf{Z}}$ verifies

$$\overrightarrow{\mathbf{p}} . \overrightarrow{\mathbf{q}} = w; \quad \overrightarrow{\mathbf{Y}}^2 = \overrightarrow{\mathbf{Z}}^2 = 1; \quad \overrightarrow{\mathbf{p}}^2 = \overrightarrow{\mathbf{q}}^2 = 0$$

$$\mathbf{p}, \mathbf{q} \perp \mathbf{Y}, \mathbf{Z}; \quad \mathbf{Y} \perp \mathbf{Z}$$

.

Now let us see how are the tensorial components of **A**.

We settle in the tensor **A** and his components.

In the mix form the matrix of tensor components ( or what is the same thing, the matrix of associated endomorphism components) is

$$(\mathbf{A}_\alpha{}^\beta) = \begin{pmatrix} \sigma\lambda & 0 & 0 & 0 \\ 0 & -\sigma\lambda & 0 & 0 \\ 0 & 0 & 0 & +\nu\mu \\ 0 & 0 & -\nu\mu & 0 \end{pmatrix} ; \alpha, \beta = 0, 3; \sigma = \pm 1; \nu = \pm 1 \tag{18}$$

A dyadic representation ( in the context of the mix components) is

$$A = \sigma\lambda(\overrightarrow{\mathbf{p}} \otimes \overrightarrow{\mathbf{p}} - \overrightarrow{\mathbf{q}} \otimes \overrightarrow{\mathbf{q}}) + \nu\mu \overrightarrow{Y} \wedge \overrightarrow{\mathbf{Z}}, \sigma = \pm 1; \nu = \pm 1 \tag{19}$$

In the covariant form the matrix of the tensor components is

$$(\mathbf{A}_{\alpha\beta}) = \begin{pmatrix} 0 & \sigma\lambda w & 0 & 0 \\ -\sigma\lambda w & 0 & 0 & 0 \\ 0 & 0 & 0 & +\nu\mu \\ 0 & 0 & -\nu\mu & 0 \end{pmatrix} ; \alpha, \beta = 0, 3; \sigma = \pm 1, \nu = \pm 1, \tag{20}$$

The physical covariant dyadic representation of the tensor is

$$\mathbf{A} = \sigma\lambda w \overrightarrow{\mathbf{p}} \wedge \overrightarrow{\mathbf{q}} + \nu\mu \overrightarrow{\mathbf{Y}} \wedge \overrightarrow{\mathbf{Z}} ; \sigma = \pm 1; \nu \pm 1 \tag{21}$$

For the electromagnetic field it is similar to [4] if $\sigma = \nu = 1$ and $w = 1$ .

The matrix of contravariant components of the tensor **A** is

$$(\mathbf{A}^{\alpha\beta}) = \begin{pmatrix} 0 & -\sigma\frac{1}{w}\lambda & 0 & 0 \\ \sigma\frac{1}{w}\lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & +\nu\mu \\ 0 & 0 & -\nu\mu & 0 \end{pmatrix} ; \alpha, \beta = 0, 3; \sigma = \pm 1; \nu = \pm 1$$

The dyadic representation of the tensor is

$$\mathbf{A} = -\sigma\lambda\frac{1}{w} \overrightarrow{\mathbf{p}} \wedge \overrightarrow{\mathbf{q}} + \nu\mu \overrightarrow{\mathbf{Y}} \wedge \overrightarrow{\mathbf{Z}} ; \sigma = \pm 1; \nu \pm 1 \tag{22}$$

It should be notice how w appears only into the context of covariant and contravariant components.

### 5.3 Reference frame: $(\vec{\mathbf{U}}, \vec{\mathbf{X}}, \vec{\mathbf{Y}}, \vec{\mathbf{Z}})$

Herein we are dealing with a reference frame associated to observer and with the invariant subspaces of the endomorphism A (see Annex B).

This reference frame $(\overrightarrow{\mathbf{U}}, \overrightarrow{\mathbf{X}}, \overrightarrow{\mathbf{Y}}, \overrightarrow{\mathbf{Z}})$ constitute an orthonormal base.

In this reference frame the metric tensor is

$$(\mathbf{g}^{\alpha\beta}) = (\mathbf{g}_{\alpha\beta}) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} ; \alpha, \beta = 0, 3$$

Without going into details the matrix of mix components of tensor **A** is

$$(\mathbf{A}_{\alpha}{}^{\beta}) = \begin{pmatrix} 0 & \sigma\lambda & 0 & 0 \\ \sigma\lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & +\nu\mu \\ 0 & 0 & -\nu\mu & 0 \end{pmatrix}; \alpha, \beta = 0, 3; \sigma = \pm 1; \nu \pm 1 \tag{23}$$

In the dyadic depiction of the tensor (in this case only for mix tensor field components) we have

$$\mathbf{A} = (\sigma\lambda(\overrightarrow{\mathbf{U}} \otimes \overrightarrow{\mathbf{X}} + \overrightarrow{\mathbf{X}} \otimes \overrightarrow{\mathbf{U}}) + \nu\mu\overrightarrow{\mathbf{Y}} \wedge \overrightarrow{\mathbf{Z}}); \sigma = \pm 1; \nu \pm 1 \tag{24}$$

This result is agreeing with [5] for $\sigma = 1; \nu = 1$

Also without going into details the matrix of covariant components of tensor $\mathbf{A}$ is

$$(\mathbf{A}_{\alpha\beta}) = \begin{pmatrix} 0 & \sigma\lambda & 0 & 0 \\ -\sigma\lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & +\nu\mu \\ 0 & 0 & -\nu\mu & 0 \end{pmatrix}; \alpha, \beta = 0, 3; \sigma = \pm 1; \nu \pm 1 \tag{25}$$

In the dyadic depiction of the tensor (in this case only for covariant tensor field components) we have

$$\mathbf{A} = \sigma\lambda\overrightarrow{\mathbf{U}} \wedge \overrightarrow{\mathbf{X}} + \nu\mu\overrightarrow{\mathbf{Y}} \wedge \overrightarrow{\mathbf{Z}}; \sigma = \pm 1; \nu \pm 1 \tag{26}$$

These results are similar to [9].

# 6 Some questions about interpretation in regard to $A$

In this section we analyze aside and in short some relevant point of interest.

## 6.1 Concept of limit for $\lambda$ and $\mu$ that is to say $\lambda \to 0$ or and $\mu \to 0$

.

In this article $\lambda$ and $\mu$ only take values on the basis of classification of $P(A)$ and afterwards on the classification of electromagnetic field. In this article it is not included the concept of limit unless otherwise specified.

### 6.2 Special significance of $w$

$w$ is the scalar product of eigenvectors of skew-adjoint endomorphism A, $\overrightarrow{\mathbf{p}}$ and $\overrightarrow{\mathbf{q}}$.

If $w = 1$ we are dealing with the Sach´s tetrad. In general **we achieve the standpoint of special relativity if** $w = 1$.

w only appears in covariant and contravariant components of $\mathbf{A}$ in the reference frame $(\vec{\mathbf{p}}, \vec{\mathbf{q}}, \vec{\mathbf{Y}}, \vec{\mathbf{Z}})$. Any way it has not impact in stress-energy tensor in any referential frame.

### 6.3 Freedom grades of $A$

Skew-adjoint endomorphisms and tensors on $L_4$ have six freedom grades. This endomorphism depends heavily on peculiarities of vectors $\overrightarrow{\mathbf{p}}$ ,$\overrightarrow{\mathbf{q}}$ , $\overrightarrow{\mathbf{Y}}$ , $\overrightarrow{\mathbf{Z}}$. Taking into account their modulus ( $\overrightarrow{\mathbf{p}}^2 = 0$ and $\overrightarrow{\mathbf{q}}^2 = 0$, $\overrightarrow{\mathbf{Y}}^2 = 1$ and $\overrightarrow{\mathbf{Z}}^2 = 1$) and the relation among them, it is not hard to see that the tensor $\mathbf{A}$ has six freedom grades.

In the case (widely agreed) in which

$\overrightarrow{\mathbf{Y}} \wedge \overrightarrow{\mathbf{Z}} = -(\overrightarrow{\mathbf{p}} \wedge \overrightarrow{\mathbf{q}})^*$

there are only five freedom grades.

## 7 Singular cases.

Until now we have analyzed fields that satisfy requirements 13. In this section we sketch some singular relevant cases.

A thorough study of these cases gains in complexity and goes out of our purposes.

Furthermore, in order that the singular cases are radiation in regard to propagation, they have to fulfil requirements that differential field equations compel. Really it is not suitable to call radiation fields to the singular cases since they can or cannot coincide with propagation fields cases. We use the term radiation according to some authors.

### 7.1 Classification of singular cases regarding annihilating and annihilating minimal polynomials.

The possible cases are in the next table:

| *Annihilating.pol.* | *Minimal.pol.* |
|---|---|
| $A.A.A^2$ | $A.A^2$ |
| $A^2.A^2$ | $A^2$ |
| $A.A^3$ | $A.A^3$ |

All these cases deserve a detailed study aside. They are beyond the scope of this article.

## 8 Electromagnetic tensor

Henceforth the skew-adjoint tensor A that matches electromagnetic field will be denoted **F**.

The values of $\sigma$ and $\nu$ are in some extent arbitrary. Simply $\sigma$ and $\nu$, as a matter of fact, will not be required in our classification hereafter. In general we choice $\sigma = 1$ , $\nu = 1$.

The types of electromagnetic field we show in this article agree with 13. We call *pure fields* to these electromagnetic field. Pure electromagnetic field is the case that concern us in this article.

The electromagnetic tensor field with covariant components 21 in the dyadic context is :

$$\mathbf{F} = w\lambda \overrightarrow{\mathbf{p}} \wedge \overrightarrow{\mathbf{q}} + \mu \overrightarrow{\mathbf{Y}} \wedge \overrightarrow{\mathbf{Z}} \tag{27}$$

The electromagnetic tensor field with contravariant components 22 in the dyadic context is :

$$\mathbf{F} = -\lambda \frac{1}{w} \overrightarrow{\mathbf{p}} \wedge \overrightarrow{\mathbf{q}} + \mu \overrightarrow{\mathbf{Y}} \wedge \overrightarrow{\mathbf{Z}} \tag{28}$$

In these cases it is suitable point out $\lambda \neq 0$ or $\mu \neq 0$ and $\lambda \neq \mu$

## 9 Stress-energy tensor of electromagnetic field

In this section we are only concerned with stress-energy tensor of *pure electromagnetic field*

As we know, in this case the stress-energy tensor is

$$\mathbf{T} = -(\mathbf{F}.\mathbf{F} - \frac{1}{4}\mathbf{G}.tr(\mathbf{F}.\mathbf{F}))$$

We start in the context of mix components matrix of **F** according with 18 ( being $\sigma = 1$ , $\nu = 1$ ) with the base $(\mathbf{p}, \mathbf{q}, \mathbf{Y}, \mathbf{Z})$. Firstly we construct **F.F** and tr**(F.F)**.

Therefore

$$(\mathbf{F}_\alpha{}^\beta) = \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & -\lambda & 0 & 0 \\ 0 & 0 & 0 & \mu \\ 0 & 0 & -\mu & 0 \end{pmatrix} ; \alpha, \beta = 0, 3$$

Then

$$(\mathbf{F}.\mathbf{F})_\alpha{}^\beta = \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & -\lambda & 0 & 0 \\ 0 & 0 & 0 & \mu \\ 0 & 0 & -\mu & 0 \end{pmatrix} \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & -\lambda & 0 & 0 \\ 0 & 0 & 0 & \mu \\ 0 & 0 & -\mu & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \lambda^2 & 0 & 0 & 0 \\ 0 & \lambda^2 & 0 & 0 \\ 0 & 0 & -\mu^2 & 0 \\ 0 & 0 & 0 & -\mu^2 \end{pmatrix}$$

; $\alpha, \beta = 0, 3$

tr$(\mathbf{F}.\mathbf{F})$=$2(\lambda^2 - \mu^2)$

Taking into account that the matrix of mix components of $\mathbf{G}$ is

$$(\mathbf{G}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

it is not hard to see that

$$(\mathbf{T}_\alpha{}^\beta) = \begin{pmatrix} -\chi^2 & 0 & 0 & 0 \\ 0 & -\chi^2 & 0 & 0 \\ 0 & 0 & \chi^2 & 0 \\ 0 & 0 & 0 & \chi^2 \end{pmatrix}$$

where $\chi^2 = \frac{1}{2}(\lambda^2 + \mu^2)$

To go from a pseudo-orthonormal base (see Annex **B**) $(\overrightarrow{\mathbf{p}}, \overrightarrow{\mathbf{q}}, \overrightarrow{\mathbf{Y}}, \overrightarrow{\mathbf{Z}})$ to the orthonormal base $(\overrightarrow{\mathbf{U}}, \overrightarrow{\mathbf{X}}, \overrightarrow{\mathbf{Y}}, \overrightarrow{\mathbf{Z}})$ we utilize the equations

$$\overrightarrow{\mathbf{p}} = a(\overrightarrow{\mathbf{X}} + \overrightarrow{\mathbf{U}})$$
$$\overrightarrow{\mathbf{q}} = b(\overrightarrow{\mathbf{X}} - \overrightarrow{\mathbf{U}})$$

Here $\overrightarrow{\mathbf{p}}.\overrightarrow{\mathbf{q}} = w$ where $w = 2ab$, $a$ and $b$ are components of null eigenvectors in the reference frame $(\overrightarrow{\mathbf{U}},\overrightarrow{\mathbf{X}})$ ;

$\overrightarrow{\mathbf{U}}^2 = -1, \overrightarrow{\mathbf{Y}}^2 = \overrightarrow{\mathbf{Z}}^2 = 1$

$\overrightarrow{\mathbf{X}}^2 = 1$ and $\overrightarrow{\mathbf{p}}.\overrightarrow{\mathbf{Y}} = \overrightarrow{\mathbf{p}}.\overrightarrow{\mathbf{Z}} = \overrightarrow{\mathbf{q}}.\overrightarrow{\mathbf{Y}} = \overrightarrow{\mathbf{q}}.\overrightarrow{\mathbf{Z}} = \overrightarrow{\mathbf{X}}.\overrightarrow{\mathbf{Y}} = \overrightarrow{\mathbf{X}}.\overrightarrow{\mathbf{Z}} = \overrightarrow{\mathbf{Y}}.\overrightarrow{\mathbf{Z}} = \overrightarrow{\mathbf{U}}.\overrightarrow{\mathbf{X}} = \overrightarrow{\mathbf{U}}.\overrightarrow{\mathbf{Y}} = \overrightarrow{\mathbf{U}}.\overrightarrow{\mathbf{Z}} = 0$.

Therefore for an observer with a reference frame $(\overrightarrow{\mathbf{U}}, \overrightarrow{\mathbf{X}}, \overrightarrow{\mathbf{Y}}, \overrightarrow{\mathbf{Z}})$, taking into account the previous change of base, and in the context of covariants and contravariant components, it is easy to infer without going into details that the stress-energy tensor is:

$$(\mathbf{T}_{\alpha\beta}) = (\mathbf{T}^{\alpha\beta}) = \begin{pmatrix} \chi^2 & 0 & 0 & 0 \\ 0 & -\chi^2 & 0 & 0 \\ 0 & 0 & \chi^2 & 0 \\ 0 & 0 & 0 & \chi^2 \end{pmatrix}$$

Showing dyadic form in the context of covariant and contravariant components then we have

$$T = \chi^2(\mathbf{u} \otimes \mathbf{u} - \mathbf{z} \otimes \mathbf{z} + \mathbf{x} \otimes \mathbf{x} + \mathbf{y} \otimes \mathbf{y}) \tag{29}$$

where $\chi^2 = \frac{1}{2}(\lambda^2 + \mu^2)$ is the ***proper energy*** (see [4]).

## 10 Conclusions

In this article we have brought out a formulation of electromagnetic field resting upon the analysis of annihilating polynomial of skew-adjoint endomorphism.

Thereby the electromagnetic tensor of a pure electromagnetic field in a covariant form and in the reference frame $(\overrightarrow{\mathbf{p}}, \overrightarrow{\mathbf{q}}, \overrightarrow{\mathbf{Y}}, \overrightarrow{\mathbf{Z}})$ is :

$$\mathbf{F} = (\lambda w)\overrightarrow{\mathbf{p}} \wedge \overrightarrow{\mathbf{q}} + \mu \overrightarrow{\mathbf{Y}} \wedge \overrightarrow{\mathbf{Z}}$$

where

$$\overrightarrow{\mathbf{p}}.\overrightarrow{\mathbf{q}} = w; \quad \overrightarrow{\mathbf{Y}}^2 = \overrightarrow{\mathbf{Z}}^2 = 1; \quad \overrightarrow{\mathbf{p}}^2 = \overrightarrow{\mathbf{q}}^2 = 0$$

$$\overrightarrow{\mathbf{p}}, \overrightarrow{\mathbf{q}} \perp \overrightarrow{\mathbf{Y}}, \overrightarrow{\mathbf{Z}}; \quad \overrightarrow{\mathbf{Y}} \perp \overrightarrow{\mathbf{Z}}$$

If in the observer material reference frame $(\overrightarrow{\mathbf{p}}, \overrightarrow{\mathbf{q}}, \overrightarrow{\mathbf{Y}}, \overrightarrow{\mathbf{Z}})$ is w=1 then we are in the basic theory. If $w \neq 1$ the factor $w$ appears in the electromagnetic tensor in his covariant and cotravariant form in the cited reference frame.

In the reference frame $(\overrightarrow{\mathbf{U}}, \overrightarrow{\mathbf{X}}, \overrightarrow{\mathbf{Y}}, \overrightarrow{\mathbf{Z}})$ the w factor does not appears.

ANNEXES

# A Some propositions about lorentzian vectorial spaces geometry.

In this article some propositions and definitions necessaries to deal with vectorial spacetime, are shown. Most of these propositions are shown without proving since it is not the subject of this article.

As far as I know the most thorough study about lorentzian vectorial space is in [1], [2], and [11] .

$L_n$ stands for a lorentzian space of signature (-1,1,....1). In the spacetime $L_4$ we are limited to dimension 4, signature (-1,1,1,1).

$E_i \quad i = 1, 2, 3$ stands for an euclidian subspace of $L_4$.

$I_1$ stands for a null straight line.

$I_k \quad k = 1, 2, 3, 4$ stands for a null subspace or space.

## A.1 Definitions

***Spacelike***

It is easily checked that a subspace generated by **orthogonal** spacelike vectors , is an euclidian subspace. This euclidian subspace is named *spacelike subspace.*

All his vectors are spacelike.

A subspace spacelike is euclidian.

***Causal subspace***

It contains timelike, spacelike and null vectors.

***Null subspace***

It is a subspace formed of a null vector and a subspace orthogonal to it. Vectors of this orthogonal space are *spacelike.*

If the subspace orthogonal to the null vector is a 3D spacelike , then this is constituted by null vector and three spacelike vectors ortogonal to it. This space is called *properly null space* $I_4$

A *properly null space* is solely formed by one null vector and also a subspace generated by three spacelike vectors orthogonal to the null vector. It does not contains timelike vectors.

Vectors orthogonal to a causal subspace, are spacelike vectors. Thereby a subspace orthogonal to a null vector is spacelike.

In a null subspace, a reference frame formed by spacelike, is moving at light speed.

## A.2 Orthogonality relations

Two timelike vectors can not be orthogonal.

If two null vectors are orthogonal then they are proportional. They are orthogonal to themselves.

A vector orthogonal to a timelike vector is spacelike vector. A vector orthogonal to a null vector, is spacelike vector or null vector.

A vector orthogonal to a spacelike vector is spacelike or timelike or null vector.

# B Concepts about referential frame at rest respect an observer.

## B.1 Definition of reference frame at rest respect an observer.

We define a reference frame at rest related to an observer, as reference frame such that the matrix $(\mathbf{G_r})$ of the metric tensor covariant or contravariant components in this reference frame is:

$$(\mathbf{G_r}) = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

Thereby the vectorial base of the reference frame at rest respect an observer must be orthonormal.

## B.2 Passage from the pseudo-orthonormal base to an orthonormal base at rest.

That is from the pseudo-orthonormal base $(\overrightarrow{\mathbf{p}}, \overrightarrow{\mathbf{q}}, \overrightarrow{\mathbf{Y}}, \overrightarrow{\mathbf{Z}})$ to the orthonormal base $(\overrightarrow{\mathbf{U}}, \overrightarrow{\mathbf{X}}, \overrightarrow{\mathbf{Y}}, \overrightarrow{\mathbf{Z}})$

As we saw earlier in a pseudo-orthonormal base we have:

$$\overrightarrow{\mathbf{p}}^2 = \overrightarrow{\mathbf{q}}^2 = 0; \overrightarrow{\mathbf{p}}.\overrightarrow{\mathbf{q}} = w; \overrightarrow{\mathbf{Y}}.\overrightarrow{\mathbf{Z}} = \overrightarrow{\mathbf{p}}.\overrightarrow{\mathbf{Y}} = \overrightarrow{\mathbf{p}}.\overrightarrow{\mathbf{Z}} = \overrightarrow{\mathbf{q}}.\overrightarrow{\mathbf{Y}} = \overrightarrow{\mathbf{q}}.\overrightarrow{\mathbf{Z}} = 0$$

$$\overrightarrow{\mathbf{Y}}^2 = \overrightarrow{\mathbf{Z}}^2 = 1$$

In the orthonormal basis $(\overrightarrow{\mathbf{U}}, \overrightarrow{\mathbf{X}}, \overrightarrow{\mathbf{Y}}, \overrightarrow{\mathbf{Z}})$ we have

$$\overrightarrow{\mathbf{U}}^2 = -1; \overrightarrow{\mathbf{X}}^2 = \overrightarrow{\mathbf{Y}}^2 = \overrightarrow{\mathbf{Z}}^2 = 1$$
$$\overrightarrow{\mathbf{U}}.\overrightarrow{\mathbf{X}} = \overrightarrow{\mathbf{U}}.\overrightarrow{\mathbf{Y}} = \overrightarrow{\mathbf{U}}.\overrightarrow{\mathbf{Z}} = \overrightarrow{\mathbf{X}}.\overrightarrow{\mathbf{Y}} = \overrightarrow{\mathbf{X}}.\overrightarrow{\mathbf{Z}} = \overrightarrow{\mathbf{Y}}.\overrightarrow{\mathbf{Z}} = 0$$

The passage equations are:

$$\overrightarrow{\mathbf{p}} = a_p\overrightarrow{\mathbf{X}} + b_p\overrightarrow{\mathbf{U}}$$

$$\overrightarrow{\mathbf{q}} = a_q\overrightarrow{\mathbf{X}} + b_q\overrightarrow{\mathbf{U}}$$

$\overrightarrow{\mathbf{Y}}$ and $\overrightarrow{\mathbf{Z}}$ remain the same.

It is easily checked that:

$$a_p = \varepsilon b_p = a$$

$$a_q = \eta b_q = b$$

It must be $\eta = -\varepsilon$. Then the transition equations become:

$$\mathbf{p} = a(\varepsilon\overrightarrow{\mathbf{U}} + \overrightarrow{\mathbf{X}})$$

$$\mathbf{q} = b(-\varepsilon\overrightarrow{\mathbf{U}} + \overrightarrow{\mathbf{X}})$$

Here we have $\overrightarrow{\mathbf{p}}.\overrightarrow{\mathbf{q}} = 2ab = w$.

To keep the orientation toward the future must be $a > 0$, $b < 0$, $\varepsilon = +1$, thereby $w = 2ab < 0$

Then

$$\mathbf{p} = a(\overrightarrow{\mathbf{U}} + \overrightarrow{\mathbf{X}})$$

$$\mathbf{q} = b(-\overrightarrow{\mathbf{U}} + \overrightarrow{\mathbf{X}})$$

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